SEMINAR ÜBER ALGEBRAISCHE FLÄCHEN
SEMINAR ON ALGEBRAIC SURFACES

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Das Thema das Seminars sind *algebraische Flächen* (darunter verstehen wir im Seminar glatte projektive Varietäten von Dimension 2 über einem algebraisch abgeschlossenen Körper. Während die Einschränkung auf Dimension 2 im Vergleich zur allgemeinen Situation vieles erleichtert, ist die Situation schon wesentlich komplizierter als für algebraische Kurven, d.h. für 1-dimensionale Varietäten.

Wir werden einen Schwerpunkt darauf legen, einige interessante und wichtige Beispielklassen von algebraischen Flächen kennenzulernen.

Im wesentlichen folgen wir Kapitel V von Hartshornes Buch *Algebraic Geometry.* In den ersten Vorträgen sollen einige allgemeine Ergebnisse der algebraischen Geometrie diskutiert werden, die im folgenden benötigt werden. Weitere Referenzen sind in der Bibliographie angegeben.

We will study algebraic surfaces, more precisely smooth projective varieties of dimension 2 over an algebraically closed field. While the restriction to dimension 2 simplifies many things in comparison to the general theory, the situation is already considerably more complicated than for algebraic curves, i.e., 1-dimensional varieties.

We will focus particularly on understanding some interesting and important classes of examples of algebraic surfaces.

Mostly we will follow Chapter V of Hartshorne's Algebraic Geometry. In the first few talks we will discuss some general results from algebraic geometry which we will use later on. See the bibliography for further references.

Anforderungen/Vorkenntnisse: Gute Kenntnisse in Algebraischer Geometrie, etwa im Umfang der Vorlesungen Algebraische Geometrie I und II aus dem WS 2014/15 und SS 2015. Der Stoff aus [H] II, 1–8, III, 1–5, 8, IV.1 sollte alles umfassen, was benötigt wird. Die Motivation, sich ein mathematisches Thema anzueignen.

Organisatorisches: Der Vortrag soll an der Tafel gehalten werden und nicht länger als 80 Minuten dauern. Danach soll sich in einer kurzen Feedback-Runde jeder der Zuhörer kurz zu Stärken und Schwächen des Vortrags äußern.

Nach Wahl der/des Vortragenden kann der Vortrag auf Deutsch oder auf Englisch gehalten werden. Upon the choice of the speaker, the talk can be given in German or in English.

Program

Note to the speakers: In most talks you will not be able to present all details, so you should make sure to plan ahead what exactly you want to present, and what to omit. This is true in particular for the first few talks which partly will have a flavor of an overview talk. Usually, it is best to *not omit examples*!

Regardless of this remark, as a first step you should make sure that you yourself do understand all the details (including *exercises* and *details left to the reader*). Probably there will be some questions which you won't be able to answer yourself, so you should get in touch with Barinder Banwait early on to straighten everything out.

1. **Divisors and line bundles, ampleness.** An important tool to study a surface will be looking at the curves contained in it, or more precisely at *divisors* on the surface.

Content of the talk:

- Give a brief "reminder" on *Weil divisors* versus *Cartier divisors* versus *line bundles* on a noetherian locally factorial integral scheme. ([H], II.6, but obviously you will have to shorten this quite a lot; cf. [GW] Ch. 11)
- Ample and very ample line bundles, [H] II.7 up to Lemma 7.8 (cf. [GW] Ch. 13)
- Ampleness and cohomology, [H] III.5, Thm. 5.2 and Prop. 5.3
- Mention also Bertini's Theorem ([H] II.8.18)

2. **Serre duality.** While duality for curves was covered in Jochen Heinloth's course, we will need the theory —not surprisingly— also for higher-dimensional varieties.

Content of the talk: [H] III.7 up to 7.12.3. In the seminar, we will need the theory only for smooth varieties, so you may and should make this assumption, either throughout or whenever it simplifies the presentation. In particular, make sure to state the main result in this case (i.e., the combination of loc. cit., Cor. 7.7 and Cor. 7.12).

Further References: Compare [L] 6.4 and the references given there.

3. **Zariski's main theorem.** We will need (a version of) Zariski's Main Theorem [H] III.11.4, and of Stein factorization. (For other variants and other approaches, see e.g., [GW] Ch. 12 and Mumford's Red Book.)

Content of the talk: A quick "reminder" on higher direct image sheaves ([H] III.8). [H] III.11. Make sure you have enough time left to explain how the corollaries (11.2–5) follow from the theorem; if required, you may be sketchier on the proof of the theorem itself.

4. **Cohomology and base change.** As the last one of the "preliminary" talks, we will study cohomology and base change, i.e., for a morphism $f\colon X \to Y$ and a coherent $\mathscr{O}_X$-module $\mathscr{F}$, flat over $\mathscr{O}_Y$, we will investigate the relationship between the fibers $R^i f_* \mathscr{F} \otimes \kappa(y)$ of the higher direct image sheaves, and the cohomology groups $H^i(X_y, \mathscr{F}_{|X_y})$ of the fibers $X_y := X \times_Y \operatorname{Spec} \kappa(y)$ with coefficients in the restriction of $\mathscr{F}$.

Content of the talk: [H] III.12.

Further References: See the remarks at the end of III.12 in loc. cit.

5. **The intersection pairing on an algebraic surface.** Given two curves on a surface, typically the intersection will consist of finitely many points, so it is a natural idea to think about the "intersection number" of two curves, or more generally of two divisors on a surface. Obviously, to get a reasonable notion, one has to be a little careful (e.g., count points with appropriate multiplicity). In this talk we will define the intersection pairing on the divisor class group of a surface.

Content of the talk: [H] V.1 until (and including) Prop. 1.4.

Further References: [Ba] Ch. 1.

6. **The theorem of Riemann-Roch for surfaces.** Similarly as in the theory of curves, there is a Riemann-Roch theorem for surfaces (albeit with a slightly more complicated statement), and it is of similar importance for the development of the theory.

Content of the talk: [H] V.1, 1.4.1 until (and including) Cor. 1.8.

7. **The Hodge index theorem and the Nakai-Moishezon criterion.** In this talk, we look at two important applications of the theorem of Riemann-Roch.

Content of the talk: [H] V.1, Definition after Cor. 1.8 until end of chapter.

8. **Ruled surfaces I.** We now start the study of the class of *ruled surfaces*, i.e., of surfaces X with a surjective morphism $\pi\colon X \to C$ to a smooth projective curve, such that all fibers are projective lines.

The German term for *ruled surfaces* is *Regelfläche*.

Content of the talk: Explain what we need about projective bundles $\mathbb{P}(\mathscr{E})$, see [H] II.7, including the relevant exercises. Also say something about loc.cit., Ex. V.1.7. Then cover [H] V.2 until Remark 2.7.1.

9. **Ruled surfaces II.** We continue the study of ruled surfaces, in particular the relation with locally free sheaves of rank 2 on the base curve.

Content of the talk: [H] V.2, Prop. 2.8 until (and including) Cor. 2.14.

10. **Ruled surfaces III.** We continue the study of ruled surfaces and use our results to prove a theorem of Atiyah on vector bundles of rank 2 on an elliptic curve.

Content of the talk: Recall some results on elliptic curves, in particular the classification of line bundles (see [H] IV.1.3.7, IV.4). [H] V.2, Theorem 2.15 until (and including) Remark 2.16.1.

11. **Ruled surfaces IV.** To conclude our study of ruled surfaces, we will determine all ample divisors on rational ruled surfaces, and also consider this question in the general case.

Content of the talk: [H] V.2, 2.17 until the end of the chapter. You may be more sketchy on the final part (Prop. 2.20 . . .), or omit it, if required.

12. **Monoidal transformations.** A *monoidal transformation* of a surface is simply the blow-up of a single closed point. This is an important tool to produce new surfaces from a given one. We need to understand its effect on the divisors of the surface.

Content of the talk: Start with a short reminder on blow-ups ([H] I.4, II.7; [GW] Ch. 13). [H] V.3 until (and including) Prop. 3.8. If there is time, you could say a few words (and draw a picture or two) about the final part of V.3.

13. **Cubic Surfaces I.** Now we will consider *cubic surfaces*, i.e., surfaces defined as the zero set in $\mathbb{P}^3_k$ of a homogeneous polynomial of degree 3. We start by studying curves in the projective plane containing a prescribed set of points.

Content of the talk: [H] V.4 until (and including) 4.5.1.

14. **Cubic Surfaces II: The 27 Lines.** With the results of the previous talk at our disposal, we can analyze what happens if we blow up the projective plane in a number of points. For us, the most important case will be the blow up of $\mathbb{P}^2_k$ in 6 points, no three of which are collinear, and not all six lying on a conic. In this case we obtain a cubic surface. On the other hand, it can be shown that every cubic surface arises in this way, up to isomorphism. We will show that every such surface contains exactly 27 lines, a famous classical result in algebraic geometry.

Content of the talk: [H] V.4, 4.6 until the end of the chapter.

Further References: For an approach to "the 27 lines" on a cubic surface not using cohomology, see [GW] Ch. 16.

Literatur

[Ba] L. Badescu, *Algebraic Surfaces*, Springer Universitext.

[Be] A. Beauville, *Surfaces Algébriques Complexes*, Astérisque 54.

[GW] U. Görtz, T. Wedhorn, *Algebraic Geometry I*, Vieweg.

[H] R. Hartshorne, *Algebraic Geometry*, Springer GTM 52.

[L] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics 6.

[R] M. Reid, *Chapters on Algebraic Surfaces*, in: Complex algebraic geometry, IAS/Park City Mathematics Series; `http://homepages.warwick.ac.uk/ masda/surf/ParkC/` , `arxiv:alg-geom/9602006`