Algebraic Geometry IV Prof. Dr. Ulrich Görtz
SS 2024 Dr. Yingying Wang

Problem sheet 9

Due date: Wednesday, June 12

Problem 33

Let k be a perfect field of characteristic $p > 0$, and let E/k be an elliptic curve.

(1) Assume that E is supersingular, i.e., the multiplication by p map $[p]_E$ is purely inseparable (cf. Problem 32).
Show that $j(E) \in \mathbb{F}_{p^2}$.
(2) Let $E \to E'$ be a non-constant homomorphism of elliptic curves over k. Prove that E is supersingular if and only if E' is supersingular.

Problem 34

Let k be a finite field and let E/k be an elliptic curve. Assume that E is *ordinary*, i.e., E is not supersingular. Prove that $\mathrm{End}(E) \not\cong \mathbb{Z}$.

Hint. Let $q = \#k$. Then $E^{(q)} = E$, so that the relative q-Frobenius is an endomorphism of E. Use that, as we will soon prove, $\deg([n]_E) = n^2$ for all $n \in \mathbb{Z}$ (cf. Problem 20).

Problem 35

We identify the quotient $SL_2(\mathbb{Z})\backslash\mathbb{H}$ with the set of isomorphism classes of elliptic curves over $\mathbb{C}$. Show that the subset of those elliptic curves E such that $\mathrm{End}(E) \not\cong \mathbb{Z}$ is dense in $SL_2(\mathbb{Z})\backslash\mathbb{H}$ (with respect to the analytic topology).

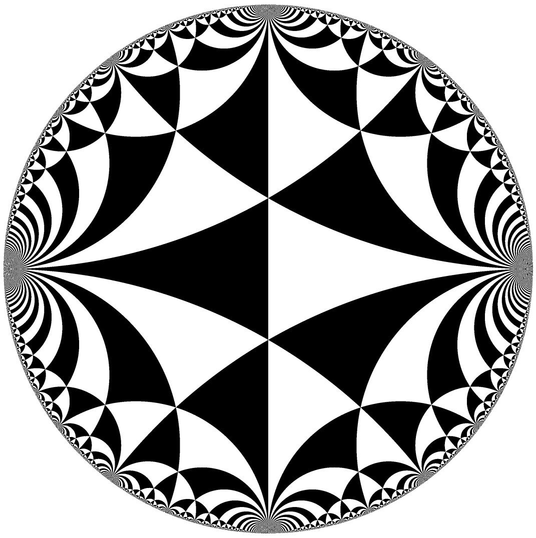
Hint. While this problem is related to Problems 27, 29, 30, you do not have to (and probably cannot) use those problems; this problem can be solved in an elementary way.

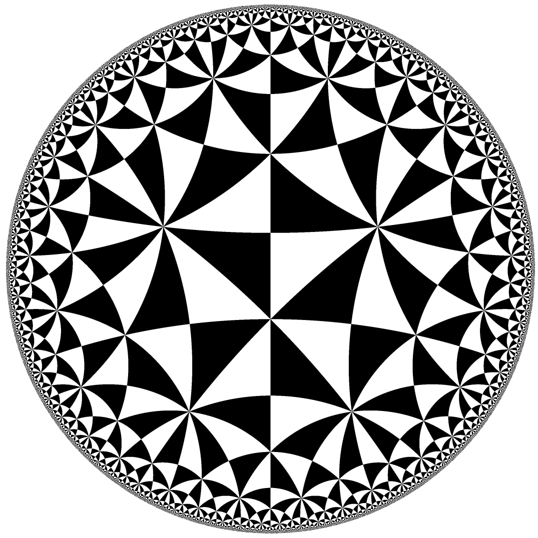
Problem 36

(1) Show that the map

$$\mathbb{H} \to D := \{z \in \mathbb{C};\ |z| < 1\}, \quad z \mapsto \frac{z-i}{z+i},$$

from the complex upper half plane to the open unit disk is an isomorphism of complex manifolds.
(2) One of the two pictures below shows (essentially) the image, under the map in (1), of the division of $\mathbb{H}$ into fundamental domains for the action of $SL_2(\mathbb{Z})$ that we discussed in class. Which one? Discuss that the other picture indicates that there exist discrete subgroups $\Gamma \subset SL_2(\mathbb{R})$ such that the quotient $\Gamma\backslash\mathbb{H}$ is compact.





Remark. For more on this topic (including the relation to Shimura varieties), see the nice and brief introduction[1] by Chao Li and the references given there.

Credits: The pictures were created with a suitably adapted version of this Python script[2].

[1] http://www.math.columbia.edu/~chaoli/docs/ShimuraCurves.html

[2] https://commons.wikimedia.org/wiki/User:Tamfang/programs