Algebraic Geometry IV Prof. Dr. Ulrich Görtz
SS 2024 Dr. Yingying Wang

Problem sheet 7

Due date: Wednesday, May 29

Problem 25

Let k be a perfect field of characteristic $p > 0$. Let C be a (smooth, projective, geometrically connected) curve over k with function field $K(C)$. As in Problem 23, let $C^{(p)}$ be the base change $C \otimes_{k,F_k} k$ with respect to the absolute Frobenius morphism of k.

(1) Show that $C^{(p)}$ is a smooth, connected curve over k. Is $C^{(p)} \cong C$?
(2) Show that the relative Frobenius $C \to C^{(p)}$ identifies $K(C^{(p)})$ with $K(C)^p$.
(3) Show that the relative Frobenius $C \to C^{(p)}$ has degree p.

Hint. For Part (3), choose a finite morphism $C \to \mathbb{P}^1_k$, and use Part (2), multiplicativity of the degree, and Problem 23 (3).

Remark. It is not essential that C be projective and *geometrically* connected. Parts (1) and (3) remain true without the assumption that k be perfect. See, e.g., [GW2] Proposition 26.71.

Problem 26

(1) Let k be a field of characteristic $\neq 2$, and let $a, b \in k$, so that the polynomial $x^3 + ax + b$ is separable. Let $d \in k^\times$. Show that the elliptic curves with affine equations $y^2 = x^3 + ax + b$ and $y^2 = x^3 + ad^2x + bd^3$ have the same j-invariant.
(2) Give an example of a (non-algebraically closed) field k and $a, b, d \in k$ such that the two curves defined as in Part (1) are not isomorphic.

Hint. Use that the j-invariant of a curve with equation $y^2 = x^3 + ax + b$ is given by $2^6 3^3 \frac{4a^3}{4a^3+27b^2}$.

Remark. One says that the curve $y^2 = x^3 + ad^2x + bd^3$ arises from the curve $y^2 = x^3 + ax + b$ by a *quadratic twist*. One can show that for $a, b \neq 0$ they are isomorphic if and only if d is a square in $k^\times$. (If one of a, b is $= 0$, there is a similar, but slightly more complicated statement.)

Problem 27

Let $K \subset \mathbb{C}$ be an imaginary quadratic number field (i.e., $K/\mathbb{Q}$ is a quadratic extension with $K \nsubseteq \mathbb{R}$). Let $\mathcal{O}_K$ be the ring of integers of K.

(1) Let $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$ be an ideal. Show that $\mathfrak{a}$ is a lattice in $\mathbb{C}$ and that $\operatorname{End}(\mathbb{C}/\mathfrak{a}) = \mathcal{O}_K$ (endomorphism ring as a complex elliptic curve).

(2) Let E be a complex elliptic curve with endomorphism ring $\mathcal{O}_K$. Show that there exists a non-zero ideal $\mathfrak{a} \subseteq \mathcal{O}_K$ such that $E \cong \mathbb{C}/\mathfrak{a}$.

Hint. For Part (1), to show $\mathrm{End}(\mathbb{C}/\mathfrak{a}) \subseteq \mathcal{O}_K$, take $\alpha \in \mathbb{C}$ with $\alpha\mathfrak{a} \subseteq \mathfrak{a}$. First show that $\alpha \in K$, then use that $\mathcal{O}_K$ is a Dedekind domain.

Problem 28

Find all lattices $\Lambda \subset \mathbb{C}$ (up to homothety, i.e., up to replacing Λ by $\gamma\Lambda$, $\gamma \in \mathbb{C}^\times$) such that the group $\mathrm{Aut}(\mathbb{C}/\Lambda)$ of all automorphisms of the elliptic curve $\mathbb{C}/\Lambda$ has more than 2 elements.

Hint. Say $\Lambda = \langle \omega_1, \omega_2 \rangle$, and $\alpha\Lambda = \Lambda$ for some $\alpha \in \mathbb{C} \smallsetminus \{1, -1\}$. View α as a change of basis of Λ and show that α is an eigenvalue of a matrix in $GL_2(\mathbb{Z})$. Then determine all possible α and Λ.

References

[GW2] U. Görtz, T. Wedhorn, *Algebraic Geometry II*, Springer Spektrum (2023).