Algebraic Geometry IV
SS 2024

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Problem sheet 5

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Problem 17

Let k be a field and let A be an abelian variety over k. Show that there is a natural identification between the tangent space $T_0(A^\vee)$ of the dual abelian variety at the neutral element and the cohomology group $H^1(A, \mathscr{O}_A)$.

Hint. Recall that $T_0 A^\vee$ can be identified with the space of k-morphisms $\operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)) \to A^\vee$ whose topological image is the point 0. Translate this into a statement about line bundles on A and use that $\operatorname{Pic}(A) = H^1(A, \mathscr{O}_A^\times)$.

Problem 18

Let k be a field. Let $f\colon E \to E'$ be a non-trivial group scheme homomorphism of elliptic curves over k. Show that for all $x, x' \in E(k)$ we have $e_x = e_{x'}$ (where e_x denotes the ramification index as in Problem 14).

Problem 19

Let k be an algebraically closed field, and let $f\colon E \to E'$ be a non-constant (group scheme) homomorphism of elliptic curves over k.

(1) Show that pullback of line bundles along f defines a group scheme homomorphism $f^*\colon \underline{\operatorname{Pic}}^0_{E'/k} \to \underline{\operatorname{Pic}}^0_{E/k}$.
(2) Denote by $\lambda\colon E \to \underline{\operatorname{Pic}}^0_{E/k}$ and $\lambda'\colon E' \to \underline{\operatorname{Pic}}^0_{E'/k}$ the natural isomorphism. The homomorphism $f^\vee := \lambda^{-1} \circ f^* \circ \lambda'\colon E' \to E$ is called the *dual isogeny* of f. Show that $f^\vee \circ f$ is the multiplication by $\deg(f)$.

Hint. For Part (2), use Problems 14 and 18 to show that $f^*(\mathscr{O}_{E'}([f(x)] - [0_{E'}])) \cong \mathscr{O}(\deg(f)([x] - [0_E]))$, and show that this implies the desired statement.

Problem 20

(1) Let k be an algebraically closed field (of characteristic $\neq 2$), let $f \in k[x]$ be a separable polynomial of degree 3 and let E be the elliptic curve given by the Weierstraß equation $y^2 = f(x)$. Determine, arguing with the geometric description of the group law on E, the points $P \in E$ such that $P + P = 0_E$. Conclude that the degree of the homomorphism $[2]\colon E \to E$, $P \mapsto 2P := P + P$, has degree 4.
(2) Show that the functor $E \mapsto \operatorname{Pic}(E)$ is *not* additive, i.e., give homomorphisms $f, g\colon E \to E'$ and a line bundle $\mathscr{L}'$ on E' such that $(f+g)^*\mathscr{L}' \not\cong f^*\mathscr{L}' \otimes g^*\mathscr{L}'$.

Hint. For Part (2), you may use that (generalizing Part (1)) the multiplication-by-m map $[m]\colon E \to E$ has degree m^2. (We will prove this in class later.) We will also prove that $E \mapsto \underline{\mathrm{Pic}}^0_{E/k}$ *is* an additive functor (so your $\mathscr{L}'$ will have to have degree $\neq 0 \dots$).