Algebraic Geometry IV Prof. Dr. Ulrich Görtz
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Problem sheet 4

Due date: Wednesday, May 8

Problem 13

Let k be a field. In this problem, a *curve* over k means a normal integral proper k-scheme of dimension 1. A *function field of curves* is an extension field K/k which is finitely generated and of transcendence degree 1.

Prove that mapping a curve C over k to its field $K(C)$ of rational functions defines an equivalence of categories between the category of curves over k with non-constant morphisms of curves as morphisms, and the category of function fields of curves over k with k-algebra homomorphisms as morphisms.

Hint. We have more or less shown that the functor is fully faithful in AG3, and you do not have to write this out. (The key ingredients are AG3 Lemma 2.12 ([GW1] Lemma 6.17, Proposition 10.52) and the valuative criterion of properness (AG3 Theorem 1.2, [GW1] Theorem 15.9), cf. the discussion following Theorem 1.2 in the AG3 lecture notes.) To show essential surjectivity, take K/k, choose an embedding $K(\mathbb{P}^1_k) \to K$ over k and define a curve C as the normalization of $\mathbb{P}^1_k$ in K (cf. Problem 9).

Remark. The construction of the hint gives a slightly different approach to showing that (normal proper) curves over k are projective than the one we discussed in the beginning of the AG3 class.

Problem 14

Let k be a field and let $f\colon C \to C'$ be a surjective morphism of (geometrically connected, smooth projective) curves. For $x \in C$ a closed point we define the *ramification index* e_x as the valuation (with respect to the discrete valuation ring $\mathscr{O}_{C,x}$) of the image (under the ring homomorphism induced by f) of a uniformizer in $\mathscr{O}_{C',f(x)}$. Furthermore, we denote by $f_x = [\kappa(x) : \kappa(f(x))]$ the degree of the residue class extension; this is called the *inertia degree*.

Let $\deg(f) = [K(C) : K(C')]$ be the degree of f. Show that for every closed point $x' \in C'$, denoting by $f^{-1}(x')$ the scheme-theoretic fiber of f over x', we have

$$\deg(f) = \dim_{\kappa(x')}(\Gamma(f^{-1}(x'), \mathscr{O}_{f^{-1}(x')}) = \sum_{x,\ f(x)=x'} e_x f_x.$$

Hint. For the first equality use that f is finite flat and hence for $V \subset C'$ affine open, $f^{-1}(V)$ is affine and $\Gamma(f^{-1}(V), \mathscr{O}_C)$ is a locally free $\Gamma(V, \mathscr{O}_{C'})$-module (of which rank?). For the second equality, as before we reduce to an affine situation, so consider a finite injective ring homomorphism $\varphi\colon A \to B$ between domains A and B such that B is a locally free A-module of rank d via φ. For a maximal ideal $\mathfrak{m} \subset A$, $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{m}))$ is a finite discrete set, so that

$$B \otimes_A \kappa(\mathfrak{m}) \cong \prod_{\mathfrak{n}} (B \otimes_A \kappa(\mathfrak{m}))_{\mathfrak{n}} \cong \prod_{\mathfrak{n}} (B_{\mathfrak{n}} \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{m})),$$

where the product ranges over all maximal ideals $\mathfrak{n} \subset B$ with $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$. Now denoting the point on C corresponding to some $\mathfrak{n}$ by x, observe that $B_{\mathfrak{n}} \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{m}) \cong B_{\mathfrak{n}}/\mathfrak{n}^{e_x} B_{\mathfrak{n}}$ and show that $\dim_{\kappa(f(x))} B_{\mathfrak{n}} \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{m}) = e_x f_x$.

Problem 15

We continue with the setting of Problem 14. Show that

(1) for every $x' \in C'(k)$, the fiber $f^{-1}(x')$ consists (as a topological space) of at most $\deg(f)$ points,
(2) if the extension $K(C)/K(C')$ is separable and k is algebraically closed, then there exists a dense open subset $V \subseteq C'$ such that for every $x' \in V(k)$ the fiber $f^{-1}(x')$ consists of exactly $\deg(f)$ points.

Hint. For Part (2) note that the assumption implies that the morphism $\operatorname{Spec}(K(C')) \to \operatorname{Spec}(K(C))$ is étale. You may use that this property "at the generic points" extends to an open neighborhood, i.e., that there exists $V \subset C'$ non-empty open such that the restriction $f_{|f^{-1}(V)}\colon f^{-1}(V) \to V$ is étale. (See [GW2] Lemma 26.10. Several results we covered in AG3 are related, e.g., Proposition 2.13 and Proposition 2.53.)

Problem 16

Let k be a field and let $f\colon C \to C'$ be a surjective morphism of (geometrically connected, smooth projective) curves. For a (Cartier) divisor D' on C' we define the pullback f^*C' as follows. Represent D' as a tuple (V_i, f_i) where $C' = \bigcup_i V_i$ is an open cover, and $f_i \in K(C')^\times$, such that $(f_i/f_j)_{V_i \cap V_j} \in \Gamma(V_i \cap V_j, \mathscr{O}_{C'})$ for all i, j. Denote by $\varphi\colon K(C') \to K(C)$ the inclusion of function fields induced by f. Then f^*D' by definition is given by the tuple $(f^{-1}(V_i), \varphi(f_i))_i$.

(1) Show that this construction is well-defined and defines a group homomorphism between the divisor groups of C' and C which maps principal divisors to principal divisors.
(2) Show that $\mathscr{O}_C(f^*D') \cong f^*\mathscr{O}_{C'}(D')$.
(3) Show that $\deg(f^*D') = \deg(f)\deg(D')$.

Hint. For (3), think of D' as a Weil divisor, reduce to the case $D' = [x']$ for some closed point x' on C' and apply Problem 14.

References

[GW1] U. Görtz, T. Wedhorn, *Algebraic Geometry I*, 2nd ed., Springer Spektrum (2020).

[GW2] U. Görtz, T. Wedhorn, *Algebraic Geometry II*, Springer Spektrum (2023).