Algebraic Geometry IV Prof. Dr. Ulrich Görtz
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Problem sheet 3

Due date: Wednesday, May 1

Problem 9

Let X be an integral scheme with function field $K = K(X)$. Let L/K be an algebraic extension. We define the *normalization* X' *of* X *in* L as follows: For $X = \operatorname{Spec}(A)$ (where A is a domain with field of fractions K) we let B be the integral closure of A in L and set $X' = \operatorname{Spec}(B)$. In general, we define the normalization by gluing affine pieces.

Show that this defines a scheme X' together with a morphism $X' \to X$, that X' is integral, $K(X') = L$, and that X' is normal (i.e., all local rings of $\mathscr{O}_{X'}$ are integrally closed in $K(X')$).

Hint. The key point is that taking the integral closure is compatible with localizations. If you need further hints, see [GW1] Section (12.11).

Problem 10

Let k be a field and let $f\colon X \to Y$ be a surjective morphism of (geometrically connected, smooth projective) curves over k. Let $\deg(f) = [K(X) : K(Y)]$ be the degree of f. Show that the morphism f is finite and flat.

Hint. For the finiteness, use that the normalization of Y in $K(X)$ is finite over Y (since Y is of finite type over a field and the extension $K(X)/K(Y)$ is finite; in general it is a subtle question whether the normalization of a scheme Z is finite over Z, see [GW1] Section (12.12)). For flatness use that a module M over a Dedekind ring R is flat if and only if it is torsion-free.

Problem 11

Let k be a field, and let E/k be an elliptic curve with neutral element $0 \in E(k)$. Set $C = E \setminus \{0\}$, an affine curve over k.

Prove that $\operatorname{Pic}(C)$ and $E(k)$ are isomorphic.

Hint. Describe $\operatorname{Pic}(C)$ in terms of divisors of E, cf. AG2, Prop. 3.17 (= [GW1] 11.42).

Problem 12

(This problem requires some knowledge on Riemann surfaces.)

Let E be an elliptic curve over the complex numbers, with neutral element $0 \in E(\mathbb{C})$. Set $C = E \setminus \{0\}$, an affine curve over $\mathbb{C}$. Denote by $\operatorname{Pic}^{\mathrm{an}}(C^{\mathrm{an}})$ the analytic Picard group of the non-compact Riemann surface C^{an}, the analytification of the curve C. By definition this is the group of isomorphism classes of line bundles in the sense of Riemann surfaces on C. Show that

$$\mathrm{Pic}^{\mathrm{an}}(C^{\mathrm{an}}) \cong H^2(C^{\mathrm{an}}, \mathbb{Z}) = 0.$$

Hint. Denote by $\mathscr{O}_{C^{\mathrm{an}}}$ the sheaf of holomorphic functions on C^{an}, i.e., the structure sheaf of the locally ringed space C^{an}. Use that $\mathrm{Pic}^{\mathrm{an}}(C^{\mathrm{an}}) \cong H^1(C^{\mathrm{an}}, \mathscr{O}_{C^{\mathrm{an}}}^\times)$ (analogously to the algebraic case) and that (cf. cohomology vanishing on affine schemes) $H^i(C^{\mathrm{an}}, \mathscr{O}_{C^{\mathrm{an}}}) = 0$ for all $i > 0$, and use the exponential sequence on C^{an}.

Remark. The GAGA theorems imply that $\mathrm{Pic}(X) \cong \mathrm{Pic}^{\mathrm{an}}(X^{\mathrm{an}})$ for every proper k-scheme X. Combining Problems 11 and 12 gives an example that shows that the properness assumption cannot be dropped.

References

[GW1] U. Görtz, T. Wedhorn, *Algebraic Geometry I*, 2nd edition, Springer Spektrum.