Problem sheet 2

Due date: Wednesday, April 24

Problem 5

(1) Let V be a complex vector space, and let $c\colon V \to V$ be an $\mathbb{R}$-linear map with $c^2 = \mathrm{id}$ and $c(zv) = \overline{z}c(v)$ for all $z \in \mathbb{C}$, $v \in V$. (Here $\overline{z}$ denotes the complex conjugate of z.) Let $V^c = \{v \in V;\ c(v) = v\}$. Show that the $\mathbb{C}$-vector space homomorphism $V^c \otimes_{\mathbb{R}} \mathbb{C} \to V$, $v \otimes z \mapsto zv$, is an isomorphism.

(2) Using (1), show that we have an equivalence of categories between the category of $\mathbb{R}$-vector spaces and the category of $\mathbb{C}$-vector spaces V together with a map $c\colon V \to V$ as in (1), given by $W \mapsto W \otimes_{\mathbb{R}} \mathbb{C}$, with $c\colon W \otimes_{\mathbb{R}} \mathbb{C} \to W \otimes_{\mathbb{R}} \mathbb{C}$ given by $w \otimes z \mapsto w \otimes \overline{z}$.

Hint. For the surjectivity in (1), note that given $v \in V$, the vectors $\frac{1}{2}(v + c(v))$ and $\frac{1}{2i}(v - c(v))$ lie in V^c.

Remark.

(1) The statement generalizes ("Galois descent of vector spaces") to arbitrary finite Galois extensions L/K, when we consider L-vector spaces V with an action $\rho\colon \mathrm{Gal}(L/K) \to GL_K(V)$ s. t. $\rho(\gamma)(zv) = \gamma(z)\rho(v)$ for all $z \in L$, $v \in V$. It can be further vastly generalized as descent of quasi-coherent modules along a faithfully flat quasi-compact morphism of schemes. See [GW1] (14.10) ff., or [BLR] 6.1.

(2) Similarly, one obtains analogous equivalences of the category of algebras ($\mathbb{R}$-algebras vs. $\mathbb{C}$-algebras together with an $\mathbb{R}$-algebra homomorphism c as in (1)), or dually of affine schemes. This generalizes to arbitrary finite Galois extensions, as well.

Problem 6

(1) Let $\mathbb{S}$ be the Deligne torus (Problem 4). We equip $A := \Gamma(\mathbb{G}_m^2, \mathscr{O}_{\mathbb{G}_m^2})$ with the map (as in Problem 5 (1)) $c\colon A \to A$ induced by the identification $\mathbb{S} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{G}_{m,\mathbb{C}}^2$ as in Problem 3. The vector space A^c is an $\mathbb{R}$-algebra (as remarked above). Prove that $\mathbb{S} = \mathrm{Spec}(A^c)$.

(2) Let V be a finite-dimensional $\mathbb{R}$-vector space. Prove that there is a one-to-one correspondence between the set of morphisms $\mathbb{S} \to GL(V)$ of algebraic group schemes over $\mathbb{R}$ and the set of Hodge structures on V.

Hint. In (2) you should use the fact that for k a field and V a k-vector space, there is a one-to-one correspondence between morphisms $\mathbb{G}_m^r \to GL(V)$ and $\mathbb{Z}^r$-gradings $V = \bigoplus_{\underline{i} \in \mathbb{Z}^r} V_{\underline{i}}$, as sketched in the class. Apply this with $k = \mathbb{C}$ and combine it with Problem 5.

Problem 7

Let K be a field. Let $F, G \in K[X, Y, Z]$ be homogeneous polynomials of positive degrees m and n, respectively. Assume that F and G are coprime. Let $C = V_+(F), D = V_+(G) \subset \mathbb{P}^2_K$. Let $\mathscr{I} \subset \mathscr{O}_{\mathbb{P}^2_K}$ be the coherent ideal sheaf of the schematic intersection $C \cap D$. Show that for a homogeneous polynomial $H \in K[X, Y, Z]$ of degree $d = m + k = n + l$, the following are equivalent:

(i) There exist homogeneous polynomials $A, B \in K[X, Y, Z]$ of degrees k and l, respectively, such that $H = AF + BG$.
(ii) The element $H \in H^0(\mathbb{P}^2_K, \mathscr{O}(d))$ lies in the subspace $H^0(\mathbb{P}^2_K, \mathscr{I}(d))$.

Hint. For the non-trivial implication (ii) $\Rightarrow$ (i), show that there is a short exact sequence

$$0 \to \mathscr{O}(r) \to \mathscr{O}(k) \oplus \mathscr{O}(l) \to \mathscr{I}(d) \to 0,$$

where $r = k - n = l - m$, and where the maps are given by

$$\eta \mapsto (\eta G, -\eta F), \quad (\xi, \psi) \mapsto F\xi + G\psi,$$

and look at its long exact cohomology sequence.

Remark. Note that the condition in (2) is a local condition, since it can be checked in the stalks at the (finitely many) points of $C \cap D$. This result is known as Max Noether's $AF + BG$ theorem.

Problem 8

(1) Let k be an algebraically closed field, let $F \in k[X, Y, Z]$ be homogeneous of degree 3 such that $V_+(F)$ is smooth, and let $G, G' \in k[X, Y, Z]$ be homogeneous of degree 3. Assume that the intersections $V_+(F) \cap V_+(G)$ and $V_+(F) \cap V_+(G')$ (which by Bézout's theorem both consist of 9 points, counted with multiplicities) have at least 8 points in common. Then they have all 9 points in common.
(2) Show that the classical geometric description of the composition law on a smooth plane cubic curve via secants and tangents satisfies the associativity law.

Hint. For (1), apply Problem 7. Let p denote the point of $V_+(F) \cap V_+(G)$ that we need to show lies in $V_+(F) \cap V_+(G')$, and let q be the point in $V_+(F) \cap V_+(G')$ that we do not know yet to lie in the other intersection. Let L be a linear polynomial so that $V_+(L)$ contains p, but not q, and write $V_+(L) \cap V_+(F) = [p] + [r] + [s]$. Set $H = LG'$, show that the assumptions in Problem 7 (ii) are satisfied so that $H = AF + BG$ for linear polynomials A, B, and show that $V_+(B)$ intersects $V_+(F)$ in q, r, s. Thus $V_+(B) = V_+(L)$ and $p = q$. For (2), apply Part (1) with F the equation of the cubic curve and G, H suitable products of linear polynomials.

References

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Springer.

[GW1] U. Görtz, T. Wedhorn, *Algebraic Geometry I*, 2nd edition, Springer Spektrum.