Algebraic Geometry IV — Prof. Dr. Ulrich Görtz
SS 2024 — Dr. Yingying Wang

Problem sheet 13

Due date: Wednesday, July 10

Problem 49

Let $N \in \mathbb{Z}_{\geqslant 3}$. Let S be a scheme and let E/S an elliptic curve. Show that every automorphism $f\colon E \xrightarrow{\cong} E$ such that $f_{|E[N]} = \mathrm{id}_{E[N]}$ is the identity of E.

Hint. Compute the degree of $f - \mathrm{id}_E$ (cf. Problems 37, 38, 43). Use that a homomorphism g with $E[N] \subseteq \mathrm{Ker}(g)$ factors through $[N]$.

Problem 50

Let $N \in \mathbb{Z}_{\geqslant 3}$. Show that the (representable) functor

$$\mathcal{M}_{\Gamma(N)}\colon (\mathrm{Sch}/\mathbb{Z}[\tfrac{1}{N}])^{\mathrm{op}} \to (\mathrm{Sets}),$$
$$S \mapsto \Big\{(E,\alpha);\ E/S \text{ ell. c.},\ \alpha\colon \underline{(\mathbb{Z}/N\mathbb{Z})^2} \xrightarrow{\cong} E[N]\Big\} / \cong,$$

satisfies the infinitesimal lifting criterion for formal smoothness, i.e., for a scheme T and a closed subscheme $T_0 \subset T$ defined by an ideal sheaf $\mathscr{I}$ with square $\mathscr{I}^2 = 0$ every pair $(E_0, \alpha_0) \in \mathcal{M}_{\Gamma(N)}(T_0)$ can be lifted, locally on T, to a pair (E,α) over T.

Problem 51

Show that the (non-representable) functor

$$\mathcal{M}\colon (\mathrm{Sch})^{\mathrm{op}} \to (\mathrm{Sets}),$$
$$S \mapsto \{E/S \text{ ell. c.}\} / \cong,$$

does not satisfy the valuative criterion for properness, i.e., prove that there exists a discrete valuation ring R and an elliptic curve E over the field of fractions of R which is not isomorphic to the base change of an elliptic curve over R.

Hint. You may use that the discriminants of two Weierstraß equations for the same elliptic curve (over some ring K) differ by multiplication by an element of the form u^{12}, $u \in K^\times$ (as can be checked using the formula for the discriminant in terms of the coefficients and using that any two Weierstraß equations for the same curve are related by a change of coordinates).

Problem 52

Let k be a field of characteristic $p > 0$.

(1) Let G/k be a group scheme, and let $F\colon G \to G^{(p)}$ the relative Frobenius morphism (cf. Problem 23). Show that $G^{(p)}$ carries a natural group scheme structure induced from the one on G and that F is a group scheme morphism.
(2) Now let μ_p be the kernel of $F\colon \mathbb{G}_m \to \mathbb{G}_m^{(p)} (= \mathbb{G}_m)$, and let α_p be the kernel of $F\colon \mathbb{G}_a \to \mathbb{G}_a^{(p)} (= \mathbb{G}_a)$. In each case, show that we obtain a finite group scheme G over k, and that the rank $\dim_k \Gamma(G, \mathscr{O}_G)$ of G is p.
(3) Compute the R-valued points of α_p and μ_p (R a k-algebra) and conclude that α_p, μ_p and the constant group scheme $\underline{\mathbb{Z}/p\mathbb{Z}}$ are pairwise non-isomorphic.