Algebraic Geometry IV Prof. Dr. Ulrich Görtz
SS 2024 Dr. Yingying Wang

Problem sheet 12

Due date: Wednesday, July 3

Problem 45

Let k be a field, $\overline{k}$ a separable closure of k, and let E be an elliptic curve over k such that $\mathrm{End}(E)$ is isomorphic to the ring of integers of an imaginary quadratic number field. Let ℓ be a prime different from $\mathrm{char}(k)$. Prove that the natural homomorphism $\rho\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell E)$ factors through the maximal abelian quotient $\mathrm{Gal}(\overline{k}/k)^{\mathrm{ab}}$.

Remark. If k is a number field, then continuing from this, one can show that for every finite place $\mathfrak{p}$ of k, the image of the inertia group $I_{\mathfrak{p}}$ under ρ is finite. Using the "criterion of Néron-Ogg-Shafarevich" it follows from this that there exist a finite extension k'/k and an elliptic curve $\mathcal{E}$ over the ring of integers $\mathcal{O}'$ of k' such that $\mathcal{E} \otimes_{\mathcal{O}'} k' \cong E \otimes_k k'$ (one says that "$E \otimes_k k'$ has everywhere good reduction").

Problem 46

Let k be an algebraically closed field of characteristic $\neq 2, 3$ and let $S = \mathbb{A}^1_k$. Show that every relative elliptic curve E/S is constant, i.e., is isomorphic to $E_0 \times_k S$ for some elliptic curve E_0/k.

Problem 47

Let S be a scheme, E/S an elliptic curve and ω a basis of $\omega_{E/S}$. Let $f\colon E \to E$ be an isomorphism such that $f^*\omega = \omega$. Show that $f = \mathrm{id}_E$.

Hint. Let $1, x, y$ be a basis of $f_*\mathcal{O}_E(3[0])$ giving rise to a Weierstraß equation of the form $y^2 = x^3 + ax + b$ for E. Prove that $1, f^*x, f^*y$ have the same properties and conclude that $f^*x = x$, $f^*y = y$.

Problem 48

Let k be a field.

(1) Let $E \subset \mathbb{P}^2_k$ be an elliptic curve given by an equation in Weierstraß form. Show that $(0:1:0)$ is an inflexion point, i.e., the tangent to E at $(0:1:0)$ intersects E with multiplicity 3.
(2) Let $E \subset \mathbb{P}^2_k$ be an elliptic curve such that the neutral element $0 \in E(k)$ is an inflexion point. Show that the set of inflexion points in $E(k)$ coincides with the set $E[3](k)$.