Problem sheet 11

Due date: Wednesday, June 26

Problem 41

Let k be a field, let $\overline{k}$ be a separable closure of k, $\Gamma = \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group, and let ℓ be a prime number.

For an elliptic curve E over k we denote by

$$T_\ell E := \lim_{\substack{\longleftarrow \\ n}} E[\ell^n](\overline{k})$$

the (ℓ-adic) Tate module of E. This construction is functorial in E. Now assume that $\ell \neq \mathrm{char}(k)$.

It follows from the results we proved on the structure of $E[\ell^n]$ that $T_\ell E$ is a free $\mathbb{Z}_\ell$-module of rank 2 (where $\mathbb{Z}_\ell$ denotes the ring of ℓ-adic integers). Explain that the action of Γ on $\overline{k}$ induces actions on each $E[\ell^n](\overline{k})$ and altogether a continuous action of Γ on $T_\ell E$.

Now let E, E' be elliptic curves over k. Prove that the natural map

$$\mathrm{Hom}_k(E, E') \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell E, T_\ell E')$$

is injective and has image inside the Γ-invariants $\mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell E, T_\ell E')^\Gamma$.

Remark. More is true: Since $\mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell E, T_\ell E')$ is a $\mathbb{Z}_\ell$-module, the above map induces a map

$$\mathrm{Hom}_k(E, E') \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell E, T_\ell E')^\Gamma.$$

With a bit more effort, it can be shown that this map also is injective for every field k. (This easily implies that $\mathrm{Hom}_k(E, E')$ is a free $\mathbb{Z}$-module of rank $\leqslant 4$.) Tate proved that it is an isomorphism, if k is finite; it is a deep theorem of Faltings that it is an isomorphism whenever k is a number field.

Problem 42

Let k be a field, ℓ a prime number different from the characteristic of k, and E/k an elliptic curve such that $\mathrm{End}(E) = \mathcal{O}_K$, the ring of integers in a quadratic imaginary number field K. From the inclusion $\mathrm{End}(E) \subset \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell E)$ (Problem 41) we obtain an $\mathcal{O}_K$-module structure on $T_\ell E$. Prove that the Tate module $T_\ell E$ is a free $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$-module of rank 1.

Hint. We know from algebraic number theory that $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is either the product of two copies of $\mathbb{Z}_\ell$ (if ℓ splits in K) or is a discrete valuation ring which is finite free over $\mathbb{Z}_\ell$ of rank 2. In one of the cases, use that $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \subseteq \mathrm{End}(T_\ell E)$, cf. the remark to Problem 41.

Problem 43

Let E be a relative elliptic curve over a connected scheme S, and let $f\colon E \to E$ be an automorphism of E.

(1) Prove that $\deg(f) = 1$ and that $\mathrm{tr}(f) \in \{-2, -1, 0, 1, 2\}$.
(2) Give examples of elliptic curves and automorphisms f with $\mathrm{tr}(f) = 0$, $\mathrm{tr}(f) = 1$ and $\mathrm{tr}(f) = 2$.

Hint. For Part (2), recall Problem 28.

Problem 44

Let k be an algebraically closed field and let S be a proper connected k-scheme. Let E/S be a relative elliptic curve. Prove that all the curves E_s, $s \in S(k)$, are isomorphic (as elliptic curves over k).

Hint. Use the j-invariant.