Algebraic Geometry IV Prof. Dr. Ulrich Görtz
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Problem sheet 1

Due date: Wednesday, April 17

Problem 1

Let $f\colon S' \to S$ be a morphism of schemes.

Let $F'\colon (\mathrm{Sch}/S')^{\mathrm{opp}} \to (\mathrm{Sets})$ be a functor. Its direct image along f is the functor

$$f_*F'\colon (\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets}), \quad T \mapsto F'(T \times_S S').$$

(1) Observe that for any S-scheme T, there is an isomorphism, functorial in T and F',
$$\mathrm{Hom}_S(T, f_*F') \to \mathrm{Hom}_{S'}(T \times_S S', F').$$

(2) Conclude that f_*F' has a natural structure of group functor, if F' is a group functor (i.e., every $F'(T)$ is equipped with a group structure, functorially in T).

Problem 2

We continue with the setting of Problem 1. Now let F' be a functor which is representable by an S'-scheme X'. An S-scheme representing the functor

$$f_*F'\colon (\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets}), \quad T \mapsto X'(T \times_S S'),$$

is called the *Weil restriction of scalars* of X' along $S' \to S$ and denoted by $\mathrm{Res}_{S'/S}(X')$, if it exists. Clearly, if the Weil restriction exists, then it is uniquely determined up to unique isomorphism.

Assume that $\varphi\colon R \to R'$ is a ring homomorphism such that R' is a finite free R-module via φ. Let $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, and let f be the morphism $S' \to S$ given by φ. Let X' be an affine S'-scheme. Prove that the Weil restriction $\mathrm{Res}_{R'/R}X' := \mathrm{Res}_{S'/S}X'$ of X' along f exists.

Hint. First do the case of affine space (of possibly infinite dimension) over S', i.e., the case where X' is the spectrum of a polynomial ring over R'. As a second step, consider closed subschemes.

Problem 3

We continue with the setting of Problem 2. Let L/K be a finite Galois field extension, let $S = \mathrm{Spec}(K)$, $S' = \mathrm{Spec}(L)$, and let $f\colon S' \to S$ be the natural morphism. Let X' be an affine L-scheme. By Problem 2, the Weil restriction $\mathrm{Res}_{L/K}X'$ exists.

(1) Prove that there is a natural isomorphism

$$(\mathrm{Res}_{L/K} X') \otimes_K L \cong \prod_{\sigma \in \mathrm{Gal}(L/K)} X'^{(\sigma)},$$

where $X'^{(\sigma)}$ denotes the twist of X' by σ, i.e., the base change $X' \otimes_{L,\sigma} L$ along the automorphism $\mathrm{Spec}(L) \to \mathrm{Spec}(L)$ given by σ^{-1}.

(2) Identify the action on the right hand side of the isomorphism in Part (1) which corresponds to the natural Galois action on the (second factor of) the left hand side.

Hint. The key point is to compute $L \otimes_K L$.

Problem 4

Now consider the Deligne torus $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$, where $\mathbb{G}_m = GL_{1,\mathbb{C}} \cong \mathrm{Spec}\, \mathbb{C}[X, X^{-1}]$ denotes the multiplicative group over $\mathbb{C}$. By the above, we have $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$.

Prove that

$$\mathbb{S} \cong \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL_2(\mathbb{R});\ a, b \in \mathbb{R} \right\},$$

where we view the right hand side as a closed subscheme of $GL_{2,\mathbb{R}}$ in the obvious way.

Is $\mathbb{S}$ isomorphic to $\mathbb{A}^2_{\mathbb{R}} \smallsetminus \{0\}$?