Relative elliptic curves

5.6.2024

Def Let S be a scheme.

(1) An <u>abelian scheme</u> over S is a smooth proper group scheme A/S with (geometrically) connected fibers.

(2) A (relative) elliptic curve over S is an abelian scheme E/S of relative dimension 1.

"<u>Example</u>" There exists no relative elliptic curve over $S = \operatorname{Spec} \mathbb{Z}$. (Tate)

There exists no abelian scheme over $S = \operatorname{Spec} \mathbb{Z}$. (Abrashkin, Fontaine)

<u>Cor</u> There does not exist a smooth proj. morphism $C \to \operatorname{Spec} \mathbb{Z}$ with geom. connected fibers of genus > 0. (Apply previous result to $\underline{\operatorname{Pic}}^0_{C/S}$.)

Rigidity

(see e.g. [GW2] (27.17), cf. [Katz-Mazur] Section (2.4))

S a scheme, $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ morph. of schemes.

Def A morphism $h: X \to Y$ of S-schemes is called <u>constant</u>, if it factors through f.

We define the foll. subset of S:

$\mathrm{Const}_S(h) := \{ s \in S;\ X_s \neq \emptyset$ and $h_s: X_s \to Y_s$ is constant $\}$

schematic fibres

Rmk In the above situation, let $S' \xrightarrow{\xi} S$ be a morphism of schemes, $h': X \times_S S' \to Y \times_S S'$.

Then $\xi^{-1}(\mathrm{Const}_S(h)) = \mathrm{Const}_{S'}(h_{S'})$.

Proposition As above, consider

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

(1) Suppose that f is proper and that $f_*\mathcal{O}_X = \mathcal{O}_S$.

Then $\mathrm{Const}_S(h) \subseteq S$ is open and $h_{\mathrm{Const}_S(h)}$ is constant.

(2) Suppose that f is proper, flat, of finite presentation and has geom. connected and geom.-reduced fibres, and that g is separated and of finite presentation.

Then $\mathrm{Const}_S(h) \subseteq S$ is open and closed and $h_{\mathrm{Const}_S(h)}$ is constant.

Proof WLOG S affine, $\mathrm{Const}_S(h) \neq \emptyset$.

(1) Let $s \in \mathrm{Const}_S(h)$, $V \subseteq Y$ an affine open nbhd of the point $f_s(X_s) \subseteq Y_s \subseteq Y$.

Then $X_s \subseteq h^{-1}(V)$, and since f closed, there exists $s \in W \subseteq S$ affine open s.t. $X_W := f^{-1}(W) \subseteq h^{-1}(V)$.

$$\rightsquigarrow \begin{tikzcd} X_W \arrow[rr, "h_W"] \arrow[dr] & & V_W \arrow[dl] \\ & W & \end{tikzcd}$$

V_W affine since S, W, V all affine

Then h_W factors as

$$X_W \to \underbrace{\mathrm{Spec}\, \underbrace{\Gamma(X_W, \mathcal{O}_X)}_{= \Gamma(W, \mathcal{O}_S)}}_{= W} \to V_W.$$

So h_W is constant.

Now the assumptions f proper + $f_*\mathcal{O}_X = \mathcal{O}_S$ imply that f is an epimorphism:

- f surjective: since f proper, $\mathrm{Im}(f)$ is closed, but if f would factor through $V(\mathfrak{a}) \subseteq S = \mathrm{Spec}(R)$, $\mathfrak{a} \neq 0$, then $f_*\mathcal{O}_X = \mathcal{O}_S$ would be an $R/\mathfrak{a}$-module which is impossible.

- $\mathcal{O}_S \to f_*\mathcal{O}_X$ is isom., so in particular injective.

In particular, $X_w \neq \emptyset$ for all $w \in W$, so $W \subseteq \mathrm{Const}_S(h)$. Thus we have shown that $\mathrm{Const}_S(h) \subseteq S$ open.

Also, for each W as above the section $t_W : W \to Y$ s.t. $h_W = t_W \circ f_W$ is unique.

By gluing obtain a section over all of $\mathrm{Const}_S(h)$.

$\rightsquigarrow h_{\mathrm{Const}_S(h)}$ constant.

The proof of (2) is more delicate and we only give the key argument. See e.g. [GW2] Prop 27.96 (2) for details

- By "standard techniques" one reduces to the case that S noetherian.

- The assumptions in (2) imply $f_* \mathcal{O}_X = \mathcal{O}_S$, hence by (1), $\mathrm{Conn}_S(h)$ is open.
  One shows that an open subset is closed if and only if it is closed under specialization, so need to show: for all $x \in \mathrm{Conn}_S(h)$, $x' \in \overline{\{x\}}$, we have $x' \in \mathrm{Conn}_S(h)$.

- Given $x, x' \in \mathrm{Conn}_S(h)$ as above, we have $f(x') \in \overline{\{f(x)\}}$, hence [...] there exist a discrete valuation ring $\mathrm{Spec}\, R$ and a morphism $\mathrm{Spec}\, R \to S$ with image $\{f(x), f(x')\}$. Applying the base change $\mathrm{Spec}\, R \to S$ we may replace S by $\mathrm{Spec}\, R$.

So now assume $S = \operatorname{Spec} R$, R a dvr.

Let $s \in S$ the closed, $\eta \in S$ the generic pt.

Need to show: if $\eta \in \mathrm{Const}_S(h)$, then $s \in \mathrm{Const}_S(h)$.

- Replacing Y by $\mathrm{Im}(h)$ and using that Y/S separated, we may assume that h surjective and Y/S proper.

By assumption, $h_\eta = t \circ f_\eta$ for some $t : \operatorname{Spec} k(\eta) \to Y$

By the valuative criterion for properness, there exists a unique $\tilde{t} : \operatorname{Spec} R \to Y$ extending t.

Claim $h = \tilde{t} \circ f$

<u>Proof of claim</u> Consider the equalizer $\mathrm{Eq}(h, \tilde{t} \circ f)$, a closed subscheme of X which contains X_η.

Since X is flat over R, the smallest closed subscheme of X containing X_η is X.

$R \to A$ flat $\rightsquigarrow$

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & A \otimes \mathrm{Frac}\, R \\ A/\mathfrak{a} \arrow[ur, dotted, "\text{factors only for } \mathfrak{a}=0"'] & \end{tikzcd}$$

<u>Cor</u>. Let S be a scheme, X, Y ab. schemes $/S$, and let $f: X \to Y$ be a morphism of S-schemes that preserves the neutral element ($f \circ e_X = e_Y$).

Then f is a group scheme homomorphism.

<u>Proof</u> Consider
$$h: X \times_S X \to Y$$
$$(x, x') \mapsto f(x) f(x') f(xx')^{-1}$$

Need to show: h factors through $e_Y: S \to Y$.

Enough: h is constant

(In fact, assume $h: X \times_S X \xrightarrow{f} S \xrightarrow{t} Y$ for some t.

$\rightsquigarrow X \xrightarrow{(e, id)} X \times X \xrightarrow{h} Y$ factors through t, also through e_Y

Since $X \xrightarrow{f} S$ faithfully flat, f is epimorphism of schemes, so $t = e_Y$.)

By the proposition, we can check that h is constant on the fibers $X_s \to Y_s$ of h, $s \in S$.

Thus we are reduced to the case of abelian varieties over a field where we have proved the result already.

From this result, we get the following corollaries in the same way as for abelian varieties over a field.

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<u>Cor</u>. Let S be a scheme.

(1) Let A, A' be abelian schemes $/S$ and let $f: A \to A'$ be a morphism of schemes.

Then there exist $a' \in A'(S)$ and a group scheme homomorphism $g: A \to A'$ such that $f = t_{a'} \circ g$.

(2) Every abelian scheme is a commutative group scheme.

(3) Let A be smooth, proper $/S$ with conn. fibers, $e \in A(S)$. Then A admits at most one group scheme structure with neutral element e.

Theorem Let S be a scheme. Let $E \xrightarrow{f} S$ be a smooth proper morphism so that all fibers of f are geom. connected curves of genus 1.

Let $e : S \to E$ be a section of f.

Then we have an isomorphism $E \to \underline{\mathrm{Pic}}^0_{E/S}$ of functors $(\mathrm{Sch}/S)^{op} \to (\mathrm{Sets})$, given (on T-valued points) by $x \mapsto \mathcal{O}_{E_T}([x]) \otimes \mathcal{O}_{E_T}([0])^{-1}$.

Proof Same as in the case $S = \mathrm{Spec}\, k$, k a field.

Example Let k be a field. The Legendre family

$$\mathcal{E} = V_+(Y^2Z - X(X-Z)(X-TZ)) \subset \underset{X,Y,Z}{\mathbb{P}^2_k} \times \underset{T}{(\mathbb{A}^1_k \setminus \{0,1\})}$$

is a relative elliptic curve over $\mathbb{A}^1_k \setminus \{0,1\}$

(Similarly: other Weierstrass equations "over a base", if they define a smooth morphism.)

Def. / Prop Let S be a scheme, and let $f : A \to A'$ be a homom. of abelian schemes$/S$. We call f an isogeny if the foll. equiv. properties are satisfied:

(i) f is surjective and $\mathrm{Ker}(f) := A \times_{A'} 0_{A'}$ is finite locally free $/S$,

(ii) f is finite and flat.

Moreover, in condition (ii), 'finite' may be replaced by 'quasi-finite' (since f is proper).

Rmk Every morphism $A \to A'$ is of finite presentation since A, A' are of finite presentation.

Proof (ii) $\Rightarrow$ (i) The properties in (ii) are stable under base change; in particular $\mathrm{Ker}(f)$ is finite lo.-fr. $/S$. Also, we can check surjectivity on fibers over $s \in S$,

and f_s is open (since flat + fp), closed (since finite) and A'_s is connected for all $s \in S$.

(i) $\Rightarrow$ (ii)

- since f is a homeomorphism, all fibers of f are isomorphic, in particular, f quasi-finite $\Rightarrow$ $\mathrm{Ker}(f)$ quasi-finite

- every surjection $f: A \to A'$ is faithfully flat
(fiber criterion for flatness $\rightsquigarrow$ WLOG $S = \mathrm{Spec}\, k$, k a field.
Then A' is integral, so ("generic flatness" [GW I]-Thm 10.84) there ex. $V \subseteq A'$ open s.t. $f^{-1}(V) \to V$ flat. Now use translations to conclude that f is flat.)

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Cor Being an isogeny is stable under base change $S' \to S$, and can be checked on fibers (over all $s \in S$).

Proof. • (ii) in the theorem is stable under base change

- "f surjective" and "$\mathrm{Ker}(f)$ quasi-finite/S" can be checked on fibers, and the proof of the theorem shows this is enough.

Cor Let S be a scheme, $f: A \to A'$ a morphism of ab. schemes $/S$.
If two of the foll. conditions are satisfied, then all three are satisfied, and f is an isogeny.

(a) f finite

(b) f surjective

(c) the relative dimensions of A and A' over S are equal

Moreover, all these properties can be checked on fibers over $s \in S$.

Proof We have seen that "everything" here can be checked on fibers $\rightsquigarrow$ WLOG S is the spectrum of a field.

Then $\dim \operatorname{Im}(f) + \dim \operatorname{Ker}(f) = \dim A$] [GW1] Cor. 14.97 or [Hartshorne] III Cor. 9.6

and f (quasi-) finite $\iff \dim \operatorname{Ker}(f) = 0$.

<u>Rmk</u> Let S be a connected scheme,

$f : E \to E'$ a homomorphism of ell. curves $/S$.

Then $f = 0$ or f is an isogeny.

(The locus where f is constant is open and closed.)

Remark (Isogenies and quotients by finite locally free subgroup schemes)

If $f: A \to A'$ is an isogeny, then we can "view" A' as the quotient $A/\mathrm{Ker}(f)$. (This can be made precise e.g. using the notion of fppf-sheaves.)

Conversely, given A/S and $K \subset A$ a subgroup scheme that is finite locally free $/S$, then there exists a quotient $A \to A/K$, i.e. there exists an isogeny $A \xrightarrow{f} A'$ with $\mathrm{Ker}(f) = K$.

References: [GW2] Ch. 27

[Stix, A course on finite flat group schemes and p-divisible groups, Lecture notes] Ch. 6.

<u>Definition.</u> (Degree of isogeny) Let S be a scheme, $f : A \to A'$ an isogeny of abelian schemes $/S$. The rank of $\mathcal{O}_{\ker(f)}$ over $\mathcal{O}_S$ is called the <u>degree</u> of the isogeny f. ($\in \mathbb{Z}$ if S connected)

We also set $\deg(0) = 0$.

<u>Theorem</u> Let S be a scheme, $N \in \mathbb{Z}$, $N \neq 0$. E/S an elliptic curve, $[N]_E : E \to E$ the mult. by N. Then $[N]_E$ is an isogeny of degree N^2.

Furthermore

(1) If N is invertible on S (i.e., in $\mathcal{O}_S$), then $[N]$ is étale, and conversely.

(2) If S is the spectrum of a separably closed field k with $\mathrm{char}(k) \nmid N$, then $E[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$.

constant group scheme

(3) If S is the spectrum of a field k with $\mathrm{char}(k) = p \mid N$, then $E[p](k) \cong (\mathbb{Z}/p\mathbb{Z})^i$ with $i \in \{0, 1\}$.

<u>Proof.</u> We first check that N is an isogeny. As we have seen, we can check this on fibers, so when S is the spectrum of a field. We then find a Weierstrass equation, so we reduce to the universal case:

$S \subseteq \mathrm{Spec}\, \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ the open subscheme when the curve $V_+(Y^2Z + a_1 XYZ + \cdots)$ is smooth.

Then S is irreducible and the locus of $s \in S$ s.t. $[N]_{E_s}$ is not an isogeny is equal to $\mathrm{Const}_S([N]_E)$. As we have seen, this locus is open in S. We want to show that it is empty.

If $\mathrm{Const}_S([N]) \neq \emptyset$, then it contains the generic point η of S. Embedding $k(\eta) \subseteq \mathbb{C}$, we find, by pullback, an elliptic curve over $\mathbb{C}$ for which $[N]$ is constant — a contradiction. We also find $\deg [N] = N^2$ since the degree of an isog. is locally constant on the base.

(1) Write
$$\begin{tikzcd} E \arrow[r, "\alpha"] & S \arrow[l, bend left, "e"] \end{tikzcd}.$$
$e \leftarrow$ zero section.

Let $T_e(E/S) = e^*\Omega^1_{E/S}$ the "relative tangent space of E/S" (also called the <u>Lie algebra</u> of the group scheme E/S, and often denoted by $\mathrm{Lie}(E/S)$).
This is a locally free $\mathcal{O}_S$-module of rank 1.

Then $T_{(e,e)}(E\times_S E/S) \cong T_e(E/S)\times T_e(E/S)$

and it follows that the morphism $T_e(E/S)\to T_e(E/S)$ induced by $[N]$ by functoriality is simply the multiplication by N. Cf. AG3 Problem 3 (1).

Thus

N invertible on $S \iff [N]$ induces <u>isomorphism</u> $T_e(E/S)\to T_e(E/S)$

$\Updownarrow$ (*) next page

the natural map $[N]^*\Omega^1_{E/S}\to\Omega^1_{E/S}$ is an isomorphism

(*) $[N]^*\Omega^1_{E/S} \to \Omega^1_{E/S}$ isom $\iff$ $\Omega^{1\vee}_{E/S} \to [N]^*\Omega^{1\vee}_{E/S}$ isom

$\iff \forall\, x\in E:\ \Omega^{1\vee}_{E/S}(x) \to \left([N]^*\Omega^{1\vee}_{E/S}\right)(x) = \Omega^{1\vee}_{E/S}\left([N](x)\right)$ isom.

Can check this after pullback to $\operatorname{Spec}\overline{\kappa(s)}$, $s\in S$
$\leadsto$ WLOG $S = \operatorname{Spec}$ (alg. closed field). Since $[N]$ is group homom., it is enough to check the fiber over $x=0$, and that is precisely $T_e(E/S)$.

We have an exact sequence

$$[N]^*\Omega_{E/S} \xrightarrow{\alpha} \Omega_{E/S} \longrightarrow \underbrace{\Omega_{[N]}}_{\text{"}\Omega_{E/E}\text{ w.r.t. }[N]\text{"}} \longrightarrow 0$$

If α is invertible, it follows that $\Omega_{[N]}=0$ and $0\to[N]^*\Omega_{E/S}\to\Omega_{E/S}\to\Omega_{[N]}\to 0$ is split exact. This implies (since E/S smooth) that $[N]$ is smooth (and hence, being of rel. dim 0, étale).

On the other hand, if $[N]$ is étale, then $\Omega_{[N]}=0$, so α above is surjective, so α_s is surjective $\forall s\in S$, and we see that $N\in\kappa(s)^\times$ $\forall s\in S$, so $N\in\mathcal{O}_S^\times$.

(2) Since k is assumed to be separably closed and $E[N]/k$ is finite étale, $E[N]$ is a constant group scheme $\underline{G}$ for some finite commutative group G (with N^2 elements).

It remains to check that $G \cong (\mathbb{Z}/N\mathbb{Z})^2$.

- <u>elementary approach</u>

Write $G = G_1 \times \dots \times G_r$ as a product of cyclic groups. Since G is annihilated by N, we have $\# G_i \mid N$ for all i.

Write $G[d]$, $G_i[d]$ for the kernel of mult-by d.

Then $\prod G_i[d] = G[d] = E[d](k)$ has d^2 elem. for all $d \mid N$.

From this it easily follows that

$G \cong (\mathbb{Z}/N\mathbb{Z})^2$.

- <u>alternative approach</u>

<u>Theorem</u> Let S a <u>connected</u> scheme, $G \to S$ a finite étale S-group scheme. Then there exists a finite étale morphism $S' \to S$ such that $G \times_S S' \to S'$ is a constant S'-group scheme.

This result can be proved using the theory of the étale fundamental group of a scheme.

References:

[Stix, A course on finite flat group schemes and p-divisible groups, Lecture notes] Ch. 7,

see also [GW2] Ch. 20,

[SGA 1].

elementary proof

- finite étale morphisms are open and closed

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$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "h"] \\ & S \end{tikzcd}$$
Then g, h finite étale $\Rightarrow$ f finite étale.

Lemma $X \xrightarrow{f} S$ finite étale of degree d. Then there exists $S' \to S$ finite étale s.t. $X \times_S S' \cong \coprod_1^d S'$ as S'-schemes.

Proof by induction on d. If $d=1$, then $X \xrightarrow{\sim} S$.

For $d>1$ consider $\Delta : X \to X \times_S X$ finite étale

$$\rightsquigarrow \quad X \times_S X \cong \underbrace{\Delta(X)}_{\substack{\cong \\ X}} \amalg Y \quad \text{for } Y = X \times_S X \setminus \Delta(X)$$

Since $Y \to X$ finite étale of degree $d-1$, can apply induction.

Prop Let S be a scheme, $G \to S$ a finite étale group scheme. Then there exists $S' \to S$ finite étale s.t. $G \times_S S' \to S'$ is a constant group scheme.

Proof By the lemma, WLOG $G \cong \coprod_{\Gamma} S$ as an S-scheme for some index set Γ.

Then $G \times_S G \cong \coprod_{\Gamma \times \Gamma} S \times_S S = \coprod_{\Gamma \times \Gamma} S$.

Since the morphisms $S \xrightarrow{e} G$, $G \times G \xrightarrow{m} G$, $G \xrightarrow{i} G$ giving the structure of group scheme on G/S are S-morphisms, and since S connected they are 'given by' morphisms

$\{*\} \to \Gamma$, $\Gamma \times \Gamma \to \Gamma$, $\Gamma \to \Gamma$, i.e.

by a group structure on Γ, and $G = \Gamma_S$, the constant group scheme attached to Γ.

(3) By definition, $E[p](k)$ is a finite commutative group with $< p^2$ elements ($\deg [p] = p^2$ but $[p]$ is not étale)

and annihilated by p. This implies the claim. Cf Problems 32, 33.

Rmk For an alternative approach, see [GW2] Ch. 27.

Dual isogeny. Let S be a scheme.

Let $A \xrightarrow{f} A'$ be a morphism of abelian schemes $/S$.

Then f induces a morphism (of functors, hence of schemes*)

$$f^\vee : (A')^\vee = \underline{\mathrm{Pic}}^0_{A'/S} \longrightarrow \underline{\mathrm{Pic}}^0_{A/S} = A^\vee.$$
$$\mathcal{L} \longmapsto f^*\mathcal{L}$$

Lemma If f is an isogeny, then $f^\vee$ is an isogeny.

Proof (for elliptic curves; see [GW2] Prop. 27.213 for the general case)

- WLOG $S = \operatorname{Spec} k$ for a field k
- then enough to show that $f^\vee$ is non-constant, but this is easy (see next proposition for more precise statement).

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* It is a (difficult) theorem that for every abelian scheme A/S the functor $\underline{\mathrm{Pic}}^0_{A/S}$ is representable

$\longrightarrow$ have the dual abelian scheme $A^\vee/S$.

See [GW2] Ch. 27 for more on this.

Proposition Let $f: E \to E'$ be an isogeny of elliptic curves over a scheme S. (Identify $E = E^{\vee}$, $E' = E'^{\vee}$, as usual)

Then we have $f^{\vee} \circ f = [\deg(f)]_E$.

Proof. We know already (Problem 19) that this holds whenever S is the spectrum of a field. Thus $\mathrm{Const}_S(f^{\vee} \circ f - [\deg(f)]_E) = S$, and the claim follows.

Corollary. Let S be a scheme and let $f: E \to E'$ be an isogeny of elliptic curves $/S$. Then $f^{\vee\vee} = f$.

Proof. We have shown that $f^{\vee} \circ f = [N]_E$, so also $f^{\vee\vee} \circ f^{\vee} = [N]_{E^{\vee}}$, and

$$f^{\vee\vee} \circ [N] = f^{\vee\vee} \circ f^{\vee} \circ f = [N] \circ f = f \circ [N]$$

Since $[N]$ is an isogeny, hence an epimorphism, we obtain $f^{\vee\vee} = f$.

Theorem The functor that attaches to an abelian scheme A/S its dual abelian scheme

$$A^\vee := \underline{\mathrm{Pic}}^0_{A/S}$$ is additive. (cf. Problem 20)

Proof for elliptic curves E/S (following [Katz-Mazur] Theorem 2.6.2)

For the general case of abelian schemes see e.g. [GW2] Rmk 27.166.

We need to show:

Let $f, g: E \to E'$ be homomorphisms of elliptic curves $/S$. Then

$$(f+g)^\vee = f^\vee + g^\vee$$

(where $f^\vee$ is the dual isogeny of f, etc.,

i.e., for $\mathcal{L} \in \underline{\mathrm{Pic}}^0_{E'/S}(T)$ that $(f+g)^*\mathcal{L} \cong f^*\mathcal{L} \otimes g^*\mathcal{L}$

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr] & E' \arrow[d] \\ & S \end{tikzcd} \qquad \begin{tikzcd} E' \times_S E \arrow[d] & \mathcal{L} \\ E \arrow[u, bend left, "(f{,}\mathrm{id})"] & \end{tikzcd}$$

on E_T up to tensoring with the pullback of a line bundle on T

We can do the base change $T \to S$ and rename T as S $\leadsto$ WLOG $T = S$.

Then want $f^* \mathcal{L} \otimes g^* \mathcal{L} \cong (f+g)^* \mathcal{L} \otimes \alpha^* \mathcal{M}$

$$\begin{array}{c} E \\ \downarrow \alpha \\ S \end{array}$$

for some $\mathcal{M}$ on S.

Must have $\mathcal{M} = 0_E^* \alpha^* \mathcal{M} = 0_E^* f^* \mathcal{L} \otimes 0_E^* g^* \mathcal{L} \otimes 0_E^* (f+g)^* \mathcal{L}^{-1}$

$$= 0_{E'}^* \mathcal{L},$$

so $\alpha^* \mathcal{M} = \alpha^* 0_{E'}^* \mathcal{L} = 0^* \mathcal{L}$

$0 : E \to E'$ trivial homom

Now we view $f, g, f+g, 0$ as E-valued pts of E'.

After the base change $E \to S$, replacing S by E and E' by $E' \times_S E$ (an elliptic curve over E), we are reduced to proving the following lemma.

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<u>Lemma</u> E/S ell. c., $\mathcal{L} \in \underline{\mathrm{Pic}}^0_{E/S}(S)$, $P, Q \in E(S)$.

Then
$$P^*\mathcal{L} \otimes 0^*\mathcal{L}^{-1} \otimes Q^*\mathcal{L} \otimes 0^*\mathcal{L}^{-1} \cong (P+Q)^*\mathcal{L} \otimes 0^*\mathcal{L}^{-1}$$

<u>Proof</u>. Note that the statement is 'well-defined', i.e. independent of the choice of a representation of $\mathcal{L}$ in $\mathrm{Pic}(E)$ (fiberwise of degree 0), i.e. indep of replacing $\mathcal{L}$ by $\mathcal{L} \otimes \alpha^*\mathcal{M}$, $\mathcal{M}$ on S, $\alpha: E \to S$.

Thus may assume $\mathcal{L} = \mathcal{O}_E([R]) \otimes \mathcal{O}_E([0])^{-1}$ for some $R \in E(S)$.

Now note $P = t_P \circ 0$ (as morphism $S \to E$), so $P^*\mathcal{L} = 0^* t_P^* \mathcal{L}$. Thus need to show

$$\mathcal{O}_S \cong 0^*\big(\underbrace{t_P^*\mathcal{L} \otimes t_Q^*\mathcal{L} \otimes t_{P+Q}^*\mathcal{L}^{-1} \otimes \mathcal{L}^{-1}}\big)$$

$$\overset{\mathcal{L}=\mathcal{O}_E([R])\otimes\mathcal{O}_E([0])^{-1}}{=} \mathcal{O}_E([R-P]) \otimes \mathcal{O}([-P])^{-1} \otimes \mathcal{O}([R-Q]) \otimes \mathcal{O}([-Q])^{-1} \otimes \mathcal{O}([R-P-Q])^{-1} \otimes \mathcal{O}([-P-Q]) \otimes \mathcal{O}([R])^{-1} \otimes \mathcal{O}([0])$$

$$= \underbrace{\mathcal{O}_E([R-P]+[R-Q]-[R-P-Q]-[R])}_{\mathcal{N}} \otimes t_R^*(\mathcal{N})^{-1}$$

But $\mathcal{N}$ has degree 0 and

$$(R-P)+(R-Q)-(R-(P+Q))-R=0,$$

hence $\mathcal{N}\cong\alpha^*\mathcal{M}$ for some $\mathcal{M}$ on S.

$$\rightsquigarrow\ 0^*(\mathcal{N}\otimes t_R^*\mathcal{N}^{-1})=0^*(\pi^*\mathcal{N}\otimes \underbrace{t_R^*\pi^*}_{=\pi^*}\mathcal{N}^{-1})=0^*\mathcal{O}_E=\mathcal{O}_S,$$

as we wanted to show.

Cor. Let A/S be an abelian scheme.

Then $[N]_A^\vee=[N]_{A^\vee}$.

Let E/S be an ell. curve, $f: E \to S$ the structure morphism.

For all $s \in S$, the fiber $E_s := E \times_S \operatorname{Spec} \kappa(s)$ is an elliptic curve $/ \kappa(s)$, in particular has genus 1.

From cohomology and base change, we get:

(1) $\mathcal{O}_S \xrightarrow{\sim} f_* \mathcal{O}_E$, $R^1 f_* \mathcal{O}_E$ is a line bundle on S.

(2) $\underbrace{f_* \Omega^1_{E/S}}_{=:\ \omega_{E/S}} \overset{(*)}{\cong} e^* \Omega^1_{E/S}$ is a line bundle on S

$\omega_{E/S}$ "Hodge bundle",

$(*)$ • $\Omega^1_{E/S} \cong f^* e^* \Omega^1_{E/S}$ since E/S group scheme

$\leadsto f_* \Omega^1_{E/S} = f_*(f^* e^* \Omega^1_{E/S} \otimes \mathcal{O}_E)$

$\underset{\text{projection formula}}{=} e^* \Omega^1_{E/S} \otimes \underbrace{f_* \mathcal{O}_E}_{\overset{(1)}{=} \mathcal{O}_S}$

$R^1 f_* \Omega^1_{E/S} \cong \mathcal{O}_S$ by Serre-Grothendieck duality

(and $R^1 f_* \mathcal{O}_E \cong \omega_{E/S}^{\vee}$)

(3) If $\mathcal{L}$ is a line bundle on E which is fiberwise of degree $d \geq 1$, then

$f_* \mathcal{L}$ is a vector bundle of rank d on S,

$R^1 f_* \mathcal{L} = 0$.

Theorem Let $S = \operatorname{Spec} R$ affine, $E \xrightarrow{f} S$ an elliptic curve. Assume that $\omega_{E/S}$ is a free $\mathcal{O}_S$-module. Then there exists a cubic polynomial $F \in R[X, Y, Z]$ in Weierstraß form such that

$$E \cong V_+(F) \quad (\subseteq \mathbb{P}^2_R).$$

Proof. We start by proving the following

Lemma For every $n \in \mathbb{Z}$, $\mathcal{O}(n[0]) / \mathcal{O}((n-1)[0]) \cong \omega_{E/S}^{\otimes -n}$

Proof. Denote by $e : S \to E$ the neutral elt. This is a closed immersion; denote by $I \subset \mathcal{O}_E$ the corresp. ideal sheaf

$$\begin{tikzcd} S \arrow[r, "e"] \arrow[dr] & E \arrow[d] \\ & S \end{tikzcd}$$

Then $\omega_{E/S} = e^* \Omega^1_{E/S} \cong \underbrace{I/I^2}$

$0 \to I/I^2 \to e^*\Omega^1_{E/S} \to \underset{\substack{\shortparallel \\ 0}}{\Omega^1_{S/S}} \to 0$

left exact since $S \xrightarrow{id} S$ smooth

Now consider

$$0 \to I \to \mathcal{O}_E \to \mathcal{O}_{e(S)} \to 0.$$

$$\xrightarrow{\quad - \otimes I^{\otimes -n} \quad} \quad 0 \to \mathcal{O}((n-1)[0]) \to \mathcal{O}(n[0]) \to \underbrace{I^{-n} \otimes \mathcal{O}_{e(S)}}_{} \to 0$$

$$I^{-n} \otimes \mathcal{O}_{e(S)} = e_* e^*(I^{-n}) = e_*\left((e^* I)^{-n}\right) = e_* \, \omega_{E/S}^{\otimes -n}.$$

$\boxed{19.6.2024}$

<u>Lemma</u> Have filtration

$$0 \subset \mathcal{O}_S = f_* \mathcal{O}_E = f_* \mathcal{O}_E([0]) \subset f_* \mathcal{O}_E(2[0]) \subset \cdots$$

with graded pieces $\mathcal{O}_S = \omega^{\otimes 0}, \; \omega^{\otimes -2}, \; \omega^{\otimes -3}, \; \ldots$

<u>Proof</u> Apply f_* to $0 \to \mathcal{O}((n-1)[0]) \to \mathcal{O}(n[0]) \to e_* \omega_{E/S}^{\otimes -n} \to 0$ and use what we know about $R^1 f_* \mathcal{O}(n[0])$.

<u>Rmk</u> Since we are assuming S affine, this filtration splits (non-canonically).

Now we use our assumption that $\omega_{E/S}$ is a free $\mathcal{O}_S$-module.

Let $\pi \in H^0(S, \omega_{E/S})$ be a generator, $\mathcal{O}_S \xrightarrow[\sim]{\pi} \omega_{E/S}$.

This also determines a generator π^i of $\omega_{E/S}^{\otimes i}$ $\forall i \in \mathbb{Z}$.

$\rightsquigarrow$ find basis $1, x$ of $f_* \mathcal{O}(2[0])$,

basis $1, x, y$ of $f_* \mathcal{O}(3[0])$

with $x \mapsto \pi^{-2}$, $y \mapsto \pi^{-3}$.

$\rightsquigarrow$ x uniquely det up to $x \mapsto x + a$

y —— " —— $y \mapsto y + bx + c$,

$a, b, c \in \Gamma(S, \mathcal{O}_S)$

$\longrightarrow$ $E \to V_+(Y^2Z + a_1XYZ + a_3YZ^2 - (X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3)) \subset \mathbb{P}^2_S$

cf the discussion for $S = \operatorname{Spec} k$, k a field

Remains to show: this is an isomorphism.

Since both sides are ell. curves $/S$, we can check this on fibers (isom = isog. of degree 1) where we know the result already.