Elliptic curves / $\mathbb{C}$

References [Silverman] Ch. VI
[Hartshorne] Ch. IV.4
[AW2] Sections (26.19), (27.J4)

[Mumford, Ab. Var.] Ch. I

Remark. "Recall" equivalence of categories

$$\left\{\begin{matrix}(\text{smooth, proj., connected}) \\ \text{alg. curves } / \mathbb{C}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{compact Riemann} \\ \text{surfaces}\end{matrix}\right\}$$

$$C \longmapsto C^{an}$$

(Riemann surface := (connected) complex manifold of dim 1)

On both sides have notions of genus, and a Riemann-Roch theorem, and these are compatible.

Prop Let X be a cpt Riemann surface of genus 1. Then $X \cong \mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$.

a lattice in a $\mathbb{C}$-vector space V is a subgp generated by an $\mathbb{R}$-basis of V

Sketch of proof.

sheaf of holomorphic differentials

Since X has genus 1, $\Omega_{X/\mathbb{C}} \cong \mathcal{O}_X$,

so there ex. a nowhere vanishing holomorphic differential ω on X.

Let $\tilde{X} \to X$ be the universal cover.

The map
$$\begin{aligned} \tilde{X} &\longrightarrow \mathbb{C} \\ x &\longmapsto \int_0^x f^*\omega \end{aligned}$$
integral indep of choice of path from 0 to x since $\tilde{X}$ simply conn.

is a covering map ([...]) and thus, $\mathbb{C}$ being simply connected, an isomorphism (of top. spaces and, being holomorphic, of cpl. manifolds).

Thus $X = \tilde{X}/\pi_1(X)$ and $\pi_1(X) \subseteq \mathrm{Aut}(\mathbb{C})$
$$= \{ z \mapsto \mu z + \lambda ; \ \mu \in \mathbb{C}^\times, \ \lambda \in \mathbb{C} \}$$

Since elts of $\pi_1(X)$ act without fix pts, must have $\mu = 1$ for them,
so $\pi_1(X) \subseteq \mathbb{C}$ as topgp.

More precisely, $\pi_1(X) \subseteq \mathbb{C}$ must be a discrete subgp, and (X being compact) hence must be a lattice.

The exponential map of a complex Lie group.

X complex Lie group (connected complex manifold with holomorphic group structure, i.o.w. a connected group object in the category of complex manifolds)

$\rightsquigarrow$ $\exp : T_e X \longrightarrow X$ holomorphic,

$\exp(0) = e \in X$ neutral element,

$(d\exp)_0 = \mathrm{id}_{T_e X}$,

compatible with Lie gp homomorphisms:

$$\begin{array}{ccc} T_e X & \xrightarrow{\exp_X} & X \\ df_e \downarrow & & \downarrow f \\ T_e Y & \xrightarrow{\exp_Y} & Y \end{array}$$

commutative.

(see e.g. [Mumford, Abelian varieties, Ch. I])

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Now assume X is a <u>compact</u> complex Lie group.

<u>Proposition</u> Every <u>compact</u> complex Lie group X is commutative.

<u>Proof</u>. Consider the holomorphic map

$$\begin{aligned} X &\longrightarrow \mathrm{Aut}_{\mathbb{C}}(T_e X) \subset \mathrm{End}_{\mathbb{C}}(T_e X) \leftarrow \text{finite-dim'l } \mathbb{C}\text{-v.s.} \\ x &\longmapsto (d\,\mathrm{Int}(x))_e \end{aligned}$$

(where $\mathrm{Int}(x) : X \to X$ is conjugation $z \mapsto x z x^{-1}$).

Since X is compact, $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$, so this map is constant, i.e.

$$(d\,\mathrm{Int}(x))_e = (d\,\mathrm{Int}(e))_e = \mathrm{id}_{T_e X} \quad \forall x \in X.$$

Since $\mathrm{Int}(x) \circ \exp = \exp \circ (d\,\mathrm{Int}(x))_e = \exp$,

we get $\mathrm{Im}(\exp) \subseteq Z(X)$, the center of X.

Because $(d\exp)_0 = \mathrm{id}$ is an isomorphism,

exp restricts to an isom. on suitable open nbhds of $0, e$.

In particular, the subgroup of X generated by $\mathrm{Im}(\exp)$ is open (hence closed) and thus equal to X (since X connected).

Prop. Let X be a compact complex Lie group then X is a complex torus, more precisely $X \cong T_0X/_{H_1(X,\mathbb{Z})}$.

Proof Since X is commutative, the exponential map $\exp: T_0X \to X$ is a homomorphism

$\rightsquigarrow X \cong T_0X/_{\mathrm{Ker}(\exp)}$.

Since $\exp|_U$ injective for a suitable open neighborhood $U \ni 0$, $\mathrm{Ker}(\exp)$ is discrete.

This shows that $\mathrm{Ker}(\exp)$ is a lattice Λ in the complex v.s. T_0X (because X cpt).

$\pi_1(X)$ abelian

Further, $\Lambda = \pi_1(X,0) = H_1(X,\mathbb{Z})$.

T_0X simply connected, Λ acts by covering maps

<u>Cor</u> Let $A/\mathbb{C}$ be an abelian variety.

(1) Then $X := A^{an}$ is a complex torus.

(2) In particular, for every $n \in \mathbb{Z}$ the multiplication by n, $[n] : A \longrightarrow A$ is finite and étale and

$$A[n] := \mathrm{Ker}([n]) \cong \left(\mathbb{Z}/n\mathbb{Z}\right)^{2\dim A}$$

as finite group schemes $/\mathbb{C}$.

<u>Remark</u> (1) By the "Lefschetz principle", $[n]_A$ is finite étale for every abelian variety A over a field of char. 0.

(2) Let k be a field of positive characteristic p.
Can show: If $p \nmid n$ then $[n]_A$ is finite and étale.
However, if $p \mid n$ then $[n]$ is finite, but not étale. We have $A[p](k) \cong (\mathbb{Z}/p)^f$ for some $f \in \{0, \dots, g\}$.

Prop. Let $X = \mathbb{C}^g/\Lambda$, $X' = \mathbb{C}^{g'}/\Lambda'$ be complex tori. The natural map

$$\{A \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^g, \mathbb{C}^{g'}) ;\ A\Lambda \subseteq \Lambda'\} \to \mathrm{Hom}_0(X, X')$$

is a bijection

$$\mathrm{Hom}_0(X,X') = \{f : X \to X' \text{ holom};\ f(0) = 0\}$$

Proof. Given A in the LHS set, we clearly obtain a comm. diagram

$$\begin{array}{ccc} \mathbb{C}^g & \xrightarrow{A} & \mathbb{C}^{g'} \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}$$

and thus an element of the RHS.

Conversely, consider a 'pointed' (i.e. $0 \mapsto 0$) holom. map $X \to X'$. Fixing $0 \in \mathbb{C}^g$, $0 \in \mathbb{C}^{g'}$ as base pts in the universal covers, it lifts uniquely to

$$\begin{array}{ccc} \mathbb{C}^g & \xrightarrow{\alpha} & \mathbb{C}^{g'} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array}.$$

We need to show that α is a linear map.

Let $\omega \in \Lambda \rightsquigarrow \mathbb{C}^g \to \mathbb{C}^{g'}$

$z \mapsto \alpha(z+\omega) - \alpha(z)$

has image in the discrete set Λ',

hence is constant.

Thus $\dfrac{\partial \alpha}{\partial z_j}(z+\omega) = \dfrac{\partial \alpha}{\partial z_j}(z) \quad \forall \omega \in \Lambda$,

i.e. $\partial\alpha/\partial z_j$ induces a holomorphic $\underbrace{\mathbb{C}^g/\Lambda}_{\text{cpt.}} \to \mathbb{C}^{g'}$

and hence is constant, for all $j = 1, \dots, g$.

Consider
$$\begin{tikzcd} \mathbb{C}^g \arrow[r, "\alpha"] \arrow[rr, bend right, "\alpha_i"'] & \mathbb{C}^{g'} \arrow[r, "p_i"] & \mathbb{C} \end{tikzcd}$$

$\rightsquigarrow \dfrac{\partial \alpha_i}{\partial z_j}$ constant $\forall i, j$

$\rightsquigarrow$ α_i has power series expansion $\sum a_{ij} z_j$

$\alpha(0)=0$ in some nbhd of 0, and hence (by the

identity theorem) on all of $\mathbb{C}^g$.

Cor (1) Every holomorphic map $X \to X'$ between complex tori which maps 0 to 0 respects the group structure.

(2) For complex tori X, X' of dim g, g', resp., $\mathrm{Hom}(X, X')$ is a free $\mathbb{Z}$-module of rank $\leq 4gg'$.

Proof (2) $\mathrm{Hom}(X, X') \subseteq \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda') \cong \mathrm{Mat}_{2g' \times 2g}(\mathbb{Z})$.

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Hodge structures

let $A = \begin{cases} \mathbb{R} \\ \mathbb{Q} \\ \mathbb{Z} \end{cases}$ "HS", "rational HS", "integral HS"

Def A pure A-Hodge structure of weight n on a real vector space V is an A-submodule

$V_0 \subset V$ with $V_0 \otimes_A \mathbb{R} = V$

together with a Hodge decomposition

$$V_{\mathbb{C}} = \bigoplus_{\substack{p,q \\ p+q=n}} V^{p,q}, \quad \overline{V^{p,q}} = V^{q,p} \ \forall p,q.$$

$\{(p,q) \, ; \, V^{pq} \neq 0\}$ type of the HS

Example V a real vector space.

HS on V of type $(-1,0),(0,-1)$

$$\Big\updownarrow \ 1:1 \qquad\qquad V_{\mathbb{C}} = \underbrace{V^+ \oplus V^-}_{\pm i\text{-eigensp. of } J_{\mathbb{C}}}, \quad V^- = \overline{V^+}$$

$$\Big\updownarrow$$

$J \in \mathrm{End}_{\mathbb{R}}(V)$ complex structure $\qquad J$

(i.e. $J^2 = -1 \iff V$ becomes

$\mathbb{C}$-v.s. with $iv := J(v)$)

Prop. We have equivalences of categories:

$$\begin{tikzcd}
\text{Integral HS of type } (-1,0),(0,-1) \text{ (with obvious notion of morphisms)} \arrow[d, "\wr"] \\
\text{cpl structures (on } \mathbb{R}\text{-v.s. } V) + \text{lattice } \Lambda \subset V \arrow[d, "\wr"] \\
\text{cpl. tori } X
\end{tikzcd}$$

$$(T_0X, H_1(X,\mathbb{Z})) \qquad (V,\Lambda)$$
$$\updownarrow \qquad\qquad\qquad \downarrow$$
$$X \qquad\qquad\qquad V/\Lambda$$

(We can fix $V = \mathbb{R}^{2g} \supset \mathbb{Z}^{2g} = \Lambda$,
and the complex structure on V varies.)

Theorem (Algebraizability of complex tori)

Let $X = V/\Lambda$ be a complex torus.

The foll. are equivalent.

(i) $X = A^{an}$ for some variety $A/\mathbb{C}$

(i') $X = A^{an}$ for some abelian variety $A/\mathbb{C}$, compatibly with the group laws

(ii) X is projective, i.e. there is a closed embedding $X \hookrightarrow (\mathbb{P}^n_{\mathbb{C}})^{an}$ of complex manifolds, for some n

(iii) X is polarizable, i.e. there exists a positive definite hermitian form H on V s.t. $\operatorname{Im}(H)(\Lambda \times \Lambda) \subseteq \mathbb{Z}$.

imaginary part

Note For H as in (iii), $E := \operatorname{Im}(H)$ determines H via
$H(z,z') = E(iz,z') + iE(z,z')$

$\leadsto$ Equiv. of categories

$$
\begin{array}{ccc}
\{\text{abelian varieties}/\mathbb{C}\} & & A \\
\downarrow & & \downarrow \\
\{\text{projective complex tori}\} & & A^{an} \\
\| & & \\
\{\text{polarizable complex tori}\} & & X \\
\downarrow & & \downarrow \\
\{\text{polarizable integral HS of type } (-1,0), (0,-1)\}. & & H_1(X,\mathbb{Z})
\end{array}
$$

Line bundles on a complex torus X

References: [Mumford, AV] Ch. I, [Birkenhake-Lange, Complex ab. var.] Ch. 2
[GW2] Section (27.54).

Say $X = \mathbb{C}^g/\Lambda$, and write $p: \mathbb{C}^g \to \mathbb{C}^g/\Lambda$.

(I) $\{\text{line bundles on } X\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\text{line bundles on } \mathbb{C}^g \\ + \text{ compatible action of } \Lambda\end{array}\right\}$

$$\mathcal{L} \longmapsto p^*\mathcal{L} + \text{natural action}$$

$$\boxed{\begin{array}{c} \lambda \in \Lambda: \quad \begin{tikzcd} \mathbb{C}^g \arrow[rr, "t_\lambda"', "z \mapsto z+\lambda"] \arrow[dr, "p"'] & & \mathbb{C}^g \arrow[dl, "p"] \\ & X & \end{tikzcd} \\ \rightsquigarrow \; t_\lambda^* p^* \mathcal{L} \xrightarrow{\sim} p^*\mathcal{L} \\ \text{canonical isom since} \\ p \circ t_\lambda = p \end{array}}$$

Now all line bundles on $\mathbb{C}^g$ are trivial
(cf. Problem 12. $H^i(\mathbb{C}^g, \mathcal{O}_{\mathbb{C}^g}) = 0 \ \forall i > 0$,
$H^2(\mathbb{C}^g, \mathbb{Z}) = 0$ since $\mathbb{C}^g$ contractible)

$\rightsquigarrow$ only need to understand the possible actions on the trivial line bundle $\mathcal{O}_{\mathbb{C}^g}$.

Get injection $\mathrm{Pic}(X) \hookrightarrow \mathrm{Map}(\Lambda, \mathrm{Aut}(\mathcal{O}_{\mathbb{C}^g}))$ (whose image consists of maps satisfying a certain cocycle condition).

Note that $\mathrm{Aut}(\mathcal{O}_{\mathbb{C}^g}) = \Gamma(\mathbb{C}_g, \mathcal{O}_{\mathbb{C}^g})^\times = \Gamma(\mathbb{C}_g, \mathcal{O}_{\mathbb{C}^g}^\times)$.

(II) exponential sequence:

$$0 \to 2\pi i \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 1$$

$$\rightsquigarrow \quad 0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to \underbrace{H^1(X,\mathcal{O}_X^\times)}_{\substack{\parallel \\ \mathrm{Pic}(X)}} \to H^2(X,\mathbb{Z}) \to \cdots$$

$$\boxed{\underbrace{H^0(X,\mathcal{O}_X)}_{\substack{\parallel \\ \mathbb{C}}} \xrightarrow{\exp} \underbrace{H^0(X,\mathcal{O}_X)^\times}_{\substack{\parallel \\ \mathbb{C}^\times}} \text{ surj.}}$$

(III) As a topol. space, $X = \mathbb{C}^g/\Lambda \cong (S^1)^{2g}$,

so $H^1(X,\mathbb{Z}) \cong \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z})$,

$H^i(X,\mathbb{Z}) \cong \Lambda^i H^1(X,\mathbb{Z})$ (isom given by cup-product, use Künneth formula)

$\rightsquigarrow H^i(X,\mathbb{Z}) \cong \left\{ \begin{array}{l} \text{alternating multilinear} \\ \text{maps } \underbrace{\Lambda \times \dots \times \Lambda}_{r} \to \mathbb{Z} \end{array} \right\}$,

in particular $H^2(X,\mathbb{Z}) = \left\{ \begin{array}{l} \text{alternating bilinear forms} \\ \Lambda \times \Lambda \to \mathbb{Z} \end{array} \right\}$

We obtain the following commutative diagram:

view elts of $\mathbb{C}_1^\times = \{z \in \mathbb{C};\ |z|=1\}$ as constant maps $\mathbb{C}^g \to \mathbb{C}_1^\times \subset \mathbb{C}^\times$

$$
\begin{tikzcd}
 & & \mathrm{Map}(\Lambda, \mathrm{Aut}(\mathbb{C}^g)) \arrow[d, phantom, "\cup"] & & \\
0 \arrow[r] & \mathrm{Hom}(\Lambda, \mathbb{C}_1^\times) \arrow[ur, hook] \arrow[r] \arrow[d, equal] & H^1(X, \mathcal{O}_X^\times) \arrow[r] \arrow[d, equal] & H^2(X, \mathbb{Z}) & E|_{\Lambda\times\Lambda} \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] \arrow[d, equal] & \mathrm{Pic}(X) \arrow[r] & \underbrace{\left\{ \begin{array}{c} H \text{ hermitian form on } \mathbb{C}^g; \\ E = \mathrm{Im}(H) \\ \text{satisfies} \\ E(\Lambda\times\Lambda) \subseteq \mathbb{Z} \end{array} \right\}}_{=:\,\mathcal{H}} \arrow[u, hook] \arrow[r] & 0 \\
 & H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) & & & H \arrow[uu, mapsto]
\end{tikzcd}
$$

Theorem (Appell–Humbert). There is an isomorphism

$$\mathrm{Pic}(X) \xleftrightarrow{1:1} \mathcal{P} = \{ (H, \alpha);\ H \in \mathcal{H},\ \alpha : \Lambda \to \mathbb{C}_1^\times,\ \alpha(\lambda+\lambda') = \exp(i\pi\, \mathrm{Im}(H)(\lambda,\lambda'))\alpha(\lambda)\alpha(\lambda')\ \forall \lambda,\lambda' \in \Lambda \}$$

where the grp structure on $\mathcal{P}$ is given by

$$(H,\alpha)(H',\alpha') = (H+H', \alpha\alpha').$$

(The map $\Lambda \to \mathrm{Aut}(\mathcal{O}_{\mathbb{C}^g}) = \Gamma(\mathbb{C}^g, \mathcal{O}_{\mathbb{C}^g})^\times$

corresponding to $\mathcal{L}(H,\alpha)$ is given by

$$\lambda \mapsto \left(z \mapsto \alpha(\lambda) \exp\left(\pi H(z,\lambda) + \frac{\pi}{2} H(\lambda,\lambda) \right) \right).)$$

<u>Theorem of Lefschetz</u> Let X be a complex torus. With notation as above,

$$\mathcal{L}(H,\alpha) \text{ ample} \iff H \text{ positive definite.}$$

This can be proved by studying the spaces of global sections of $\mathcal{L}(H,\alpha)$ "explicitly".

Another approach is via the Kodaira embedding theorem in complex geometry: every complex torus is a Kähler manifold, and a positive definite hermitian form H on V induces a Kähler form. This form is integral if and only if $\mathrm{Im}(H)(\Lambda\times\Lambda) \subseteq \mathbb{Z}$. Cf. e.g. [Huybrechts, Complex geometry] Cor. 5.3.5.

Example $X = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$, $\tau \in \mathbb{H}$, ell. curve

$\rightsquigarrow$ $H(z,z') = \dfrac{1}{\operatorname{Im}(\tau)} z\overline{z'}$ is positive definite, $\in \mathcal{H}$.

Def. Let $X = V/\Lambda$ be a complex torus.
A positive definite hermitian form H on V
s.t. $\operatorname{Im}(H)(\Lambda \times \Lambda) \subseteq \mathbb{Z}$
is called a polarization on X.

(equivalently, a polarization is

- the residue class in $\operatorname{Pic}(X)/\operatorname{Pic}^0(X)$ of an ample line bundle

- a morphism $X \to \operatorname{Pic}^0(X)$ — this is again a complex torus, the dual complex torus
  of the form $x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$
  for some ample line bundle $\mathcal{L}$.)

A polarization is called *principal*, if the form $\operatorname{Im}(H)$ is a perfect pairing $\Lambda \times \Lambda \to \mathbb{Z}$, equiv. if $X \to \operatorname{Pic}^0(X)$ isom.

The space of princ. pol. ab. varieties over $\mathbb{C}$

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Let $g \geq 1$.

We define the *Siegel upper half space*

$$\mathbb{H}_g = \{ Z \in \mathrm{Mat}_g(\mathbb{C});\ Z^t = Z,\ \underbrace{\mathrm{Im}(Z)}_{\text{take imaginary part entry by entry}} \underbrace{>}_{\text{pos. definite}} 0 \}$$

Let $J = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix} \in \mathrm{Mat}_{2g}(\mathbb{Z})$.

The *symplectic group*

$$Sp_{2g}(\mathbb{R}) = \{ M \in \mathrm{Mat}_{2g}(\mathbb{R});\ M^t J M = J \}$$

Then $Sp_{2g}(\mathbb{R})$ acts (transitively) on $\mathbb{H}_g$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Z = (aZ+b)(cZ+d)^{-1} \qquad a,b,c,d \in \mathrm{Mat}_g(\mathbb{Z})$$

Example $g=1$. $Sp_2 = SL_2$.

Theorem The map

take the imaginary part entry by entry

$$\mathbb{H}_g := \{ Z \in \mathrm{Mat}_g(\mathbb{C}) ;\ Z^t = Z,\ \mathrm{Im}(Z) \underbrace{>0}_{\text{positive definite}} \}$$

$$\downarrow$$

$$\Big[\{ (A, \lambda) \text{ principally polarized ab. var. } /\mathbb{C} \text{ of dim } g \} /\cong \Big] \mathcal{A}_g(\mathbb{C})$$

which maps Z to (A, λ) with

(a) $A = \mathbb{C}^g/\Lambda$, $\Lambda = \langle Z \mid E_g \rangle$ ← lattice generated $/\mathbb{Z}$ by columns of the matrix $(Z \mid E_g)$

(b) λ the polarization attached to

$\Lambda \times \Lambda \to \mathbb{Z}$ given by $\left(\begin{array}{c|c} & E_g \\ \hline -E_g & \end{array} \right)$

(w.r.t. the basis in (a))

induces a bijection

$$Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g \xrightarrow{1:1} \mathcal{A}_g(\mathbb{C}).$$

<u>Proof</u>. It is easy to check that the conditions on Z ensure that the alternating form on Λ gives rise to a polarization, and that the map is surjective.

Principally polarized ab. varieties $A = \mathbb{C}^g/\Lambda$, $A' = \mathbb{C}^g/\Lambda'$ with polarizations H, H' are isomorphic iff and only if there is a $\mathbb{C}$-vector space isomorphism $\mathbb{C}^g \cong \mathbb{C}^g$ identifying Λ with Λ' and compatible with the polarizations.

Writing the isom. $\Lambda \xrightarrow{\sim} \Lambda'$ (compatible with the alternating forms) w.r.t. the bases given by $(Z\ E_g)$, $(Z'\ E_g)$, gives an element of $Sp_{2g}(\mathbb{Z})$ and from this it is easy to check the claim.

Remarks (1) The action of $Sp_{2g}(\mathbb{Z})$ on $\mathbb{H}_g$ is properly discontinuous (i.e., $\forall x, y \in \mathbb{H}_g$ $\exists$ open nbhds $x \in U_x, y \in U_y \subseteq \mathbb{H}_g$: $\{\gamma \in Sp_{2g}(\mathbb{Z});\ \gamma U_x \cap U_y \neq \emptyset\}$ finite)

therefore $Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g$,

equipped with the quotient topology,

is a Hausdorff topological space.

(Key pt.: $\mathbb{H}_g = \underbrace{Sp_{2g}(\mathbb{R})}_{\text{loc. cpt}} / \underbrace{U(g)}_{\text{compact subgp}}$

$Sp_{2g}(\mathbb{R}) \curvearrowright \mathbb{H}_g$ by "Möbius transformations"

$Sp_{2g}(\mathbb{Z}) \subset Sp_{2g}(\mathbb{R})$ discrete subgp

Compare with

$$GL_{2g}(\mathbb{Z}) \backslash GL_{2g}(\mathbb{R}) / GL_g(\mathbb{C}) \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{complex tori} \\ \text{of dim } g \end{array} \right\} / \cong$$

$$M \longmapsto \mathbb{C}^g / \langle M \rangle$$

lattice gen. in $\mathbb{C}^g = \mathbb{R}^{2g}$ by columns of M

This space is very much non-Hausdorff.

See e.g. [Milne, Modular functions and modular forms] Prop 2.4, Prop. 2.5

(2) There is a unique structure of normal complex analytic space on $Sp_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$ s.t. the projection $\mathbb{H}_g \to Sp_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$ is holomorphic.

(3) The space $Sp_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$ arises as a connected component of the Shimura variety for $G = GSp_{2g}/\mathbb{Q}$, X the conj. class of $h : \mathbb{S} \to G(\mathbb{R})$, $\underset{\substack{\| \\ a+ib}}{z} \mapsto \begin{pmatrix} aE_g & bE_g \\ -bE_g & aE_g \end{pmatrix}$, $K = GSp_{2g}(\hat{\mathbb{Z}})$.

In particular, it is the analytification of a quasi-proj. variety $/\mathbb{C}$.

The case $g=1$.

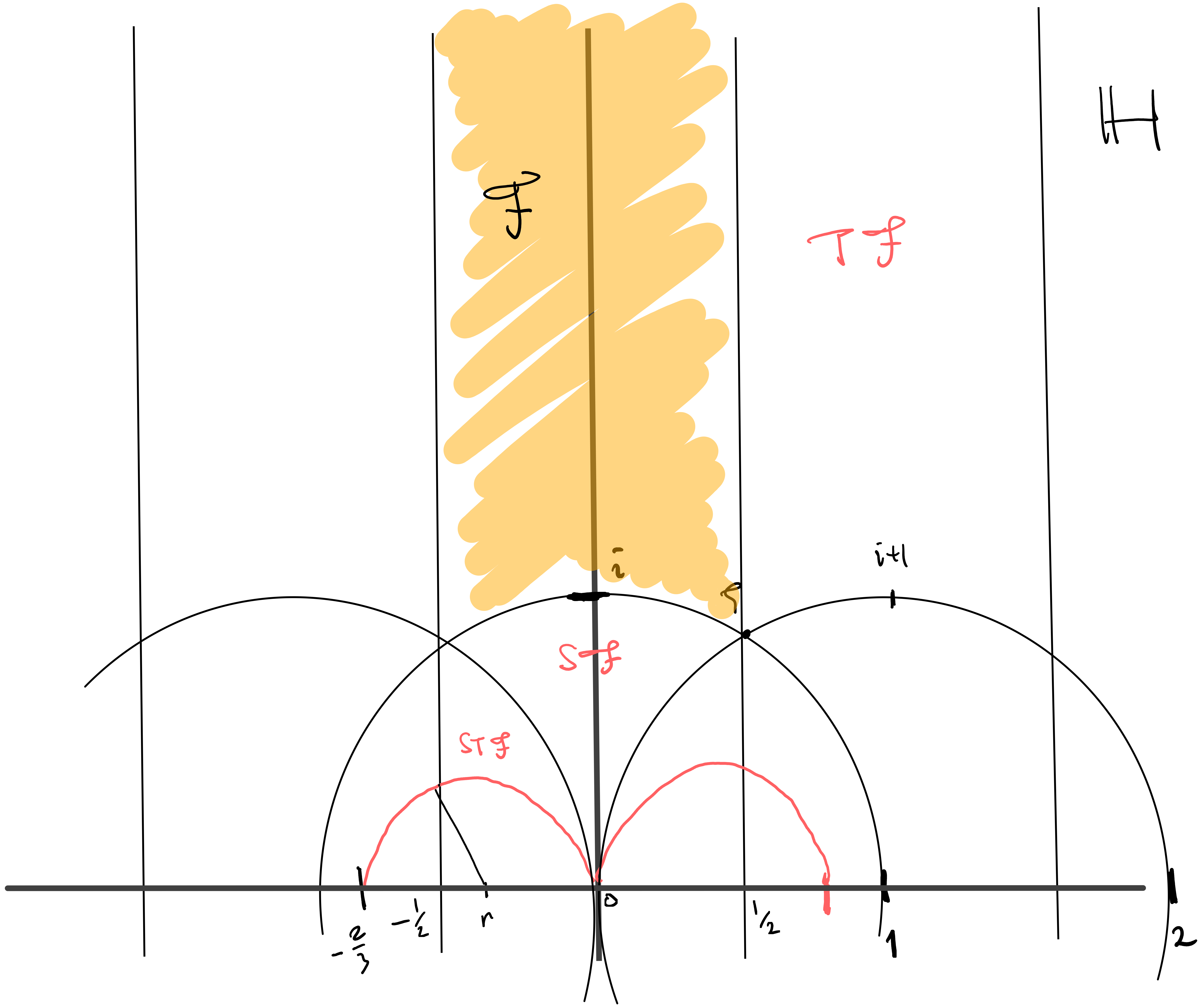
In the case $g=1$ the situation simplifies considerably.

- $\mathbb{H}_1 = \mathbb{H} = \{ z \in \mathbb{C} ; \operatorname{Im}(z) > 0 \}$ the complex upper half plane

$$\circlearrowleft$$

$$SL_2(\mathbb{R}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

$\begin{pmatrix}1&1\\&1\end{pmatrix}$, $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ generate $SL_2(\mathbb{Z})$ $\begin{pmatrix}1&1\\&1\end{pmatrix}z = z+1$, $\begin{pmatrix}0&1\\-1&0\end{pmatrix}z = -\frac{1}{z}$.



The set $\mathcal{F}$ (and thus every set of the form $g\mathcal{F}$, $g \in SL_2(\mathbb{Z})$) is a <u>fundamental domain</u> for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$:

- for all x, y in the interior of $\mathcal{F}$, $gx = y$ for some $g \in SL_2(\mathbb{Z})$ implies $x = y$
- the closure of $\mathcal{F}$ surjects onto the quotient $SL_2(\mathbb{Z})\backslash\mathbb{H}$.

- $\{\text{ell. curves}/\mathbb{C}\}/_{\cong} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{1-dim'l}\\ \text{complex tori}\end{array}\right\}/_{\cong}$

every 1-dim'l complex torus is algebraizable, by above, or using the Weierstraß $\wp$-function

$\updownarrow$ 1:1 $\sim$ every complex torus of dim 1 is polarizable

$\{\Lambda \subset \mathbb{C} \text{ lattice}\}/\mathbb{C}^\times$

$\updownarrow$ 1:1 $\quad \mathbb{Z}\oplus\mathbb{Z}\tau \leftarrow \tau$

$SL_2(\mathbb{Z})\backslash\mathbb{H}$

$$\boxed{\begin{array}{l}\mathbb{Z}\oplus\mathbb{Z}\tau = \alpha(\mathbb{Z}+\mathbb{Z}\tau'),\ \alpha\in\mathbb{C}^\times \\ \iff \begin{array}{l}\alpha = c+d\tau \\ \alpha\tau' = a+b\tau,\end{array} \quad \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbb{Z}) \\ \quad \rightsquigarrow \tau' = \dfrac{\alpha\tau'}{\alpha} = \dfrac{a+b\tau}{c+d\tau}\end{array}}$$

- $SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}$ properly discontinuous $\boxed{29.5.2024}$

since $SL_2(\mathbb{Z})$ discrete, $\mathbb{H} = SL_2(\mathbb{R}) / U(1)$

$\rightsquigarrow$ $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ Hausdorff (same for $\Gamma \backslash \mathbb{H}$ for any $\Gamma < SL_2(\mathbb{R})$ discrete subgroup)

- complex structure:

  - for $x \in \mathbb{H}$ with $\mathrm{Stab}_{SL_2(\mathbb{Z})}(x) = \{\pm 1\}$, we can choose $U \subset \mathbb{H}$ open neighborhood of x s.t. $p : \mathbb{H} \to SL_2(\mathbb{Z}) \backslash \mathbb{H}$ restricts to homeomorphism $U \to p(U)$, and define the complex structure on $p(U)$ via this homeomorphism and the complex structure on U.

  - otherwise, $\mathrm{Stab}_{SL_2(\mathbb{Z})}(x) / \{\pm 1\}$ is a finite cyclic group and one can find an open nbhd $U \subset \mathbb{H}$ of x and a comm. diagram where the horizontal arrows are homeomorphisms (and $x \mapsto 0$).

$$\begin{tikzcd} U \arrow[r] \arrow[d] & D := \{z \in \mathbb{C};\ |z|<1\} \arrow[d, "z \mapsto z^r"] \\ p(U) \arrow[r] & D \end{tikzcd}$$

$\rightsquigarrow$ define the complex structure on $p(U)$ via the lower horizontal arrows of this diagram.

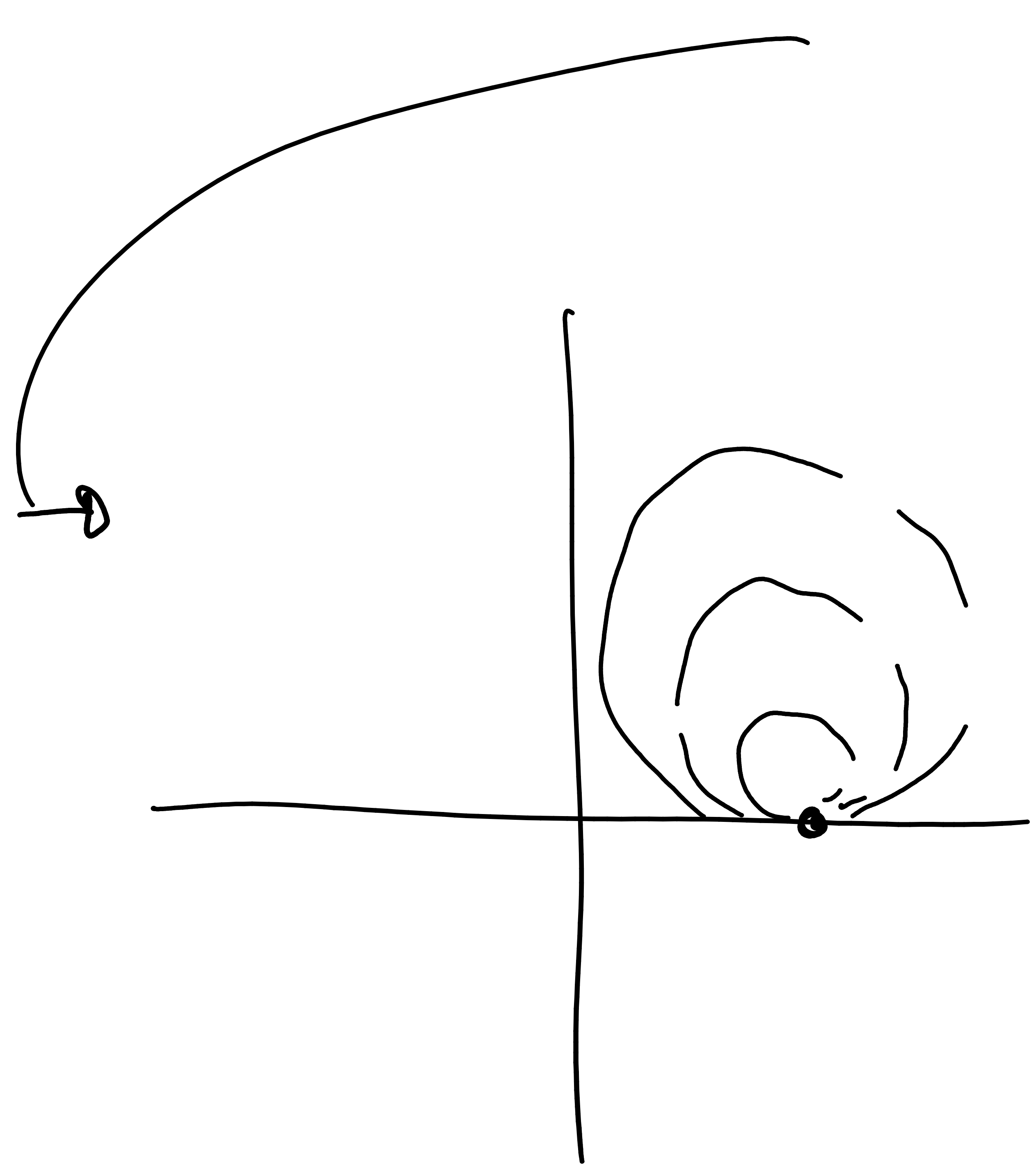
- Compactification

Let $\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$

with topology:
- on $\mathbb{H}$, take the usual topology
- basis of open nbhds of ∞:

  $U_c := \{ z \in \mathbb{C} ; \operatorname{Im}(z) > c \} \cup \{\infty\}$,

  $c \in \mathbb{R}_{>0}$
- all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ act by homeomorphisms



basis of nbhds of $a \in \mathbb{Q} \subset \mathbb{P}^1(\mathbb{Q})$:

$C \cup \{a\}$, where $C \subset \mathbb{H}$ open circle touching the real axis at a

(NB: not locally compact!)

$\longrightarrow$ $SL_2(\mathbb{Z}) \backslash \mathbb{H}^*$ compact Hausdorff

Have $SL_2(\mathbb{Z}) \circlearrowright \mathbb{P}^1(\mathbb{Q})$ transitively,

so $SL_2(\mathbb{Z})\backslash \mathbb{H}^* = SL_2(\mathbb{Z})\backslash \mathbb{H} \cup \{\infty\}$

"cusp"

Complex structure at ∞: find (cf lemma below)

$$\begin{tikzcd}
\overset{\infty}{U_c} \arrow[d] \arrow[dr, "z \mapsto \exp(2\pi i z)"] & \\
p(U_1) \arrow[r] & D_{\exp(-2\pi c)}
\end{tikzcd}$$

$c \gg 0$

disk of radius $\exp(-2\pi c)$

<u>Lemma</u> $\Gamma \circlearrowright \mathbb{H}^*$, $z \in \mathbb{H}^*$ $\leadsto$ $\exists\, U \subset \mathbb{H}^*$ open nbhd:

$$\gamma U \cap U \neq \emptyset \Rightarrow \gamma z = z.$$

(Proof by explicit computation.)

$\to$ $SL_2(\mathbb{Z})\backslash \mathbb{H}^*$ compact Riemann surface

Easy: $SL_2(\mathbb{Z})\backslash \mathbb{H}^*$ homeomorphic to the sphere,

i.e. $SL_2(\mathbb{Z})\backslash \mathbb{H}^*$ has genus 0

$\to$ $SL_2(\mathbb{Z})\backslash \mathbb{H}^* \cong \mathbb{P}^{1,an}_{\mathbb{C}}$ the Riemann sphere.

More precisely: the map

$$SL_2\backslash \mathbb{H} \to \{\text{ell. c.}/\mathbb{C}\}/_{\cong} \to \mathbb{C}$$

$$\tau \mapsto E_\tau := \mathbb{C}/_{\mathbb{Z}\oplus\mathbb{Z}\tau} \mapsto j(\tau) := j(E_\tau)$$

is holomorphic and extends to an isomorphism

$SL_2\backslash \mathbb{H}^* \xrightarrow{\sim} \mathbb{P}^{1,an}_{\mathbb{C}}$ of Riemann surfaces.

Similarly, $\Gamma \subset SL_2(\mathbb{Z})$ discrete subgroup

s.t. $\Gamma(N) := \ker\left(SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})\right) \subseteq \Gamma$ for some N

$\longrightarrow$ $\Gamma \backslash \mathbb{H}^*$ can be equipped with structure

of compact Riemann surface,

and the set of cusps, $(\Gamma\backslash\mathbb{H}^*) \setminus (\Gamma\backslash\mathbb{H})$

is finite

In particular, $\Gamma\backslash\mathbb{H}$ is the analytification

of an (affine) smooth curve $/\mathbb{C}$.

The curve $\Gamma\backslash\mathbb{H}$ can be interpreted in terms

of elliptic curves. This can be made particularly

explicit for special choices for Γ, e.g.

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) ;\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

$$\cap$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) ;\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

$$\cap$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) ;\ c \equiv 0 \bmod N \right\}$$

Example

$$Y_0(N) := \Gamma_0(N)\backslash\mathbb{H} \xleftrightarrow{1:1} \left\{ (E, G) \,;\, \begin{array}{l} E/\mathbb{C} \text{ ell. c.,} \\ G \subset E[N] \\ \text{cyclic subgp} \\ \text{of order } N \end{array} \right\} \Big/ \cong$$

$$\tau \longmapsto \Big(\mathbb{C}/_{\mathbb{Z}\oplus\tau\mathbb{Z}} \,,\ \big\langle \tfrac{1}{N} \big\rangle \Big)$$

$$\big\langle \tfrac{1}{N} \big\rangle \subset \tfrac{1}{N}(\mathbb{Z}\oplus\tau\mathbb{Z}) \big/ \mathbb{Z}\oplus\tau\mathbb{Z}$$

injective: easy

surjective: Take $E = \mathbb{C}/\Lambda$, $\Lambda \subset \mathbb{C}$,

$G \subset E[N]$ cyclic of order N,

say $G = \langle \gamma \rangle$, $\gamma \in \frac{1}{N}\Lambda$.

Then $E \cong \mathbb{C}/_{\Lambda'}$ with $\Lambda' := \dfrac{1}{N\gamma}\Lambda =: \mathbb{Z}\oplus\tau\mathbb{Z}$

(if $\mathrm{Im}(\tau) < 0$, then replace τ by $-\tau$).

Under the isom $E \cong \mathbb{C}/\Lambda'$, G is identified

with $\langle \frac{1}{N} \rangle$.

Similarly: $$Y_1(N) := \Gamma_1(N)\backslash\mathbb{H} \xleftrightarrow{1:1} \left\{ (E, P) \,;\, \begin{array}{l} E \text{ ell. c. } /\mathbb{C}, \\ P \in E[N] \\ \text{s.t. } \mathrm{ord}(P) = N \end{array} \right\} \Big/ \cong$$

Example

$$Y(N) := \Gamma(N)\backslash\mathbb{H} \xleftrightarrow{1:1} \left\{ (E,\alpha) ; \begin{array}{l} E \text{ ell. c. } / \mathbb{C} \\ \alpha : E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2 \\ \text{gp isom.} \end{array} \right.$$

such that α identifies the Weil pairing on $E[N]$ with the std alternating pairing on $(\mathbb{Z}/N\mathbb{Z})^2$ $\Big\} /\cong$

Weil pairing $E = \mathbb{C}/\Lambda$

$$e_N : E[N] \times E[N] \longrightarrow \mu_N(\mathbb{C}) = (\mathbb{Z}/N)^\times$$

$$\left(\tfrac{1}{N}\lambda, \tfrac{1}{N}\lambda'\right) \longmapsto \exp\left(\frac{2\pi i}{N} \cdot \frac{\mathrm{Im}(\bar\lambda\lambda')}{A(\Lambda)}\right)$$

$A(\Lambda)$ area of fund. parallelogram for Λ

for $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$: $A(\Lambda) = \mathrm{Im}(\tau)$,

$$e_N\left(\tfrac{1}{N}, \tfrac{\tau}{N}\right) = \exp\left(\frac{2\pi i}{N}\right)$$

$$\Gamma(N)\backslash\mathbb{H} \ni \tau \mapsto \left(E/_{\mathbb{Z}\oplus\tau\mathbb{Z}},\ \alpha : \begin{cases} \frac{1}{N} \mapsto \begin{pmatrix}1\\0\end{pmatrix} \\ \frac{\tau}{N} \mapsto \begin{pmatrix}0\\1\end{pmatrix} \end{cases} \right)$$

injection: easy

surjection: $\Gamma(N)\backslash SL_2(\mathbb{Z}) \cong SL_2(\mathbb{Z}/N\mathbb{Z})$ acts transitively on the set of ordered $\mathbb{Z}/N$-bases (b_1, b_2) of $(\mathbb{Z}/N\mathbb{Z})^2$ such that $\langle b_1, b_2 \rangle = 1 \in (\mathbb{Z}/N)^\times$

(where $\langle -,- \rangle$ denotes the std alternating pairing: $\left\langle \begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}c\\d\end{pmatrix} \right\rangle := ad - bc$)