Elliptic curves over (alg. closed) fields

References [Hartshorne] Ch. IV, [GW2] Ch. 26
[Silverman]

k a field

curve: geom. connected smooth projective k-scheme of dim 1

Def (1) An <u>abelian variety</u> over k is a geometrically connected smooth proper k-group scheme.

(2) An <u>elliptic curve</u> over k is an abelian variety over k of dimension 1.

Rmk Can show that every ab. var $/k$ is projective. (For curves we have shown this in AG3.)

Recall (AG3, Problem 19)

Let G/k be a group scheme. Then $\Omega_{G/k}$ is a free $\mathcal{O}_G$-module.

Therefore, if E/k is an elliptic curve, $\Omega_{E/k} \cong \mathcal{O}_E$ and E has genus 1. Furthermore, the neutral elt. of the group scheme structure gives us a distinguished point $O \in E(k)$.

Rigidity lemma. ([GW1] Lemma 16.55)

k a field, X/k geom. reduced, geom. conn., proper, $X(k) \neq \emptyset$.

Y/k integral, Z/k separated

$f : X \times Y \to Z$ morph. of k-schemes s.t. for some $y \in Y(k)$, $f|_{X_y}$ factors through a pt. $z \in Z(k)$.

Then f factors through $p_2 : X \times Y \to Y$

Proof Let $x \in X(k)$, $g := f \circ (x \times \mathrm{id}_Y) \circ p_2$

Want: $g = f$, i.e. $Eq(f, g) = X \times Y$

Let $z \in U \subseteq Z$ open affine.

Since X/k proper, p_2 is closed. Now $p_2^{-1}(y) \subseteq f^{-1}(U)$ by assumption, so there exists $y \in V \subseteq Y$ open s.t. $p_2^{-1}(V) \subseteq f^{-1}(U)$.

Let $y' \in V$. Obtain $f|_{X_{y'}} : X_{y'} := X \otimes_k \kappa(y') \longrightarrow U \otimes \kappa(y')$

proper $/\kappa(y')$, geom. reduced + geom conn,

affine $/\kappa(y')$

so $f|_{X_{y'}}$ factors through a $\kappa(y')$-valued pt of $U \otimes \kappa(y')$.

Hence $Eq(f,g)$ contains $X \times V$, a dense open subset of $X \times Y$. (Since Y integral, $V \subset Y$ is dense. For $U \subset X \times Y$ non-empty open, $f(X \times Y \setminus U)$ is closed, so $Y \setminus f(X \times Y \setminus U)$ intersects V. It follows that $U \cap (X \times V) \neq \emptyset$.)

Since $Eq(f,g)$ is closed in $X \times Y$ (Z being separated), and $X \times Y$ is reduced (because X is geom. reduced and Y is reduced, see [GW] Prop. 5.49), the claim follows.

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Corollary Let k be a field.

(1) Let A, B be ab. var. $/k$, and let $f: A \to B$ be a morphism of k-schemes which maps e_A to e_B. Then f is a homomorphism of group schemes.

(2) Let A be an ab. var. $/k$. Then the group law on A is commutative.

(3) The group structure on A is uniquely determined by the choice of the neutral element $0 \in A(k)$.

Proof (1) Consider

$$A \times A \to B \times B \xrightarrow{m_B} B$$
$$(a, a') \mapsto (f(aa'), (f(a)f(a'))^{-1})$$

Apply the rigidity lemma with $X = Y = A$, $Z = B$ twice to show that this morphism factors through p_1 and through p_2, and hence is constant.

(2) Apply (1) to the inverse of the group law.

(3) Follows directly from (1).

<u>Thm</u> Let E/k be a curve. The foll. are equiv:

(1) E has genus 1 and $0 \in E(k)$

(2) E carries the structure of a k-group scheme

(with neutral element $0 \in E(k)$).

From thm and rigidity, obtain equivalence of cat.: $\begin{pmatrix} \text{ell. curves} \\ /k \end{pmatrix} = \begin{pmatrix} \text{curves of genus 1} \\ \text{with a distinguished} \\ \text{pt } 0 \in E(k) \end{pmatrix}$

(2) ⇒ (1): see above

(1) ⇒ (2): (I) geometric construction,
check that this is a gp variety structure
(assoc.: see Problem Sheet 2)

(II) Show that we have an isom.

$$E \longrightarrow \underline{\mathrm{Pic}}^0_E$$

of functors $(\mathrm{Sch}/k)^{op} \to (\mathrm{Sets})$ where the right hand side is a group functor, and apply the Yoneda lemma.

<u>Proof of (1) $\Rightarrow$ (2)</u> via identification with Pic^0

E/k of genus 1 with $0 \in E(k)$ fixed point.

<u>Claim</u> Have bijection [AG2, Pbm. 32]

$$i^0 : E(k) \longrightarrow \mathrm{Pic}^0(E) := \{ \mathcal{L} \in \mathrm{Pic}(E) ;\ \deg(\mathcal{L}) = 0 \}$$

$$x \longmapsto \mathcal{O}_E([x]) \otimes \mathcal{O}_E([0])^\vee$$

Proof of claim. <u>injective</u>. $i^0(x) = i^0(y) \Rightarrow [x] \sim [y]$,

hence $x = y$ by [AG2, Pbm 50] $\left(\ell(\mathcal{O}_E([x])) = 1 \right)$

[In fact, for any curve C of genus > 0 (i.e. $\not\cong \mathbb{P}^1_k$),
$[x] \sim [y]$ implies $x = y$:
say $[x] - [y] = \mathrm{div}(f)$.
Then f defines $E \xrightarrow{f} \mathbb{P}^1_k$ surjective,
$f^*[0] = [x]$. Thus $\deg(f) = 1$, so f is isomorphism.]

<u>surjective</u>. for $\mathcal{L} \in \mathrm{Pic}^0(E)$, $\mathcal{L} \otimes \mathcal{O}_E([0])$ has
degree 1, and hence $\ell(\mathcal{L} \otimes \mathcal{O}_E([0])) = 1$. Thus
there exists a (unique) eff. divisor $[x]$ on E
with $\mathcal{L} \otimes \mathcal{O}_E([0]) = \mathcal{O}_E([x])$.

Since $\mathrm{Pic}^0(E)$ is an (abelian) group by definition, we obtain a group structure on $E(k)$.

We still need to show, however, that E is a <u>group scheme</u>, i.e. that the group structure (addition, inverse) are given by morphisms of k-schemes.

To this end, we extend the def'n of Pic^0 so as to obtain a functor $(\mathrm{Sch}/k)^0 \to (\text{Ab. Groups})$, setting $\mathrm{Pic}^0(\underbrace{E \times_{\operatorname{Spec} k} T}_{E_T}) := \{\mathcal{L} \in \mathrm{Pic}(E_T);$

$\forall t: \deg(\mathcal{L}_t) = 0$
(on $E_t := E \otimes_k k(t)$)$\}$

$$\begin{array}{ccc} E \times_k T & & E_T \\ \downarrow p & & \uparrow p^* \\ T & & \mathrm{Pic}(T) \end{array}$$

injective, since p has a section

and $\underline{\mathrm{Pic}}^0_{E/k} : (\mathrm{Sch}/k)^0 \to (\mathrm{AbGp})$, $T \mapsto \mathrm{Pic}^0(E_T)/p^*(\mathrm{Pic}(T))$.

Proposition ([GW1] Prop. 14.22)

Let $X \to S$ be a flat morphism of schemes, $Y \subseteq X$ a closed subscheme which is locally in X defined by a single equation.

Assume that for every $s \in S$, the inclusion of fibers $Y_s \subset X_s$ makes Y_s an eff. Cartier divisor in X_s (i.e. locally defined by one non-zero divisor)

If moreover S and X are locally noetherian, or if the morphism $X \to S$ is locally of finite presentation, then

Y is flat$/S$ and $Y \subset X$ is an effective Cartier divisor.

(See also [GW2] Prop. 19.28,
Cor 19.31 (1);
Compare AG3, Lemma 4.6)

effective Cartier divisor (on a general scheme):
closed subscheme defined locally by one equation given by a non-zero divisor.

See [GW1] (11.11) for the notion of (not nec. eff.) Cartier divisor on a general scheme.

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Theorem (Fiber criterion for flatness) [GW1, Thm 14.25]

Let S be a scheme,

$g: X \to S$, $h: Y \to S$ morphisms,

$f: X \to Y$ a morphism of S-schemes,

$\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module.

$$\begin{tikzcd} \mathcal{F} & & \\ X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & S & \end{tikzcd}$$

Let $x \in X$, $y = f(x)$, $s = g(x) = h(y)$ so that $\mathcal{F}_x \neq 0$.

Assume that
- S, X, Y loc. noetherian + $\mathcal{F}$ coherent

or
- g, h lfp, $\mathcal{F}$ of finite presentation

Then the following are equivalent:

(i) $\mathcal{F}$ is g-flat at x, and $\mathcal{F}_s := \mathcal{F}_{|X_s}$ is f_s-flat at x

(ii) h is flat at y and $\mathcal{F}$ is f-flat at x.

$f_s: X_s \to \operatorname{Spec} k(s)$

For the difficult direction (i) $\Rightarrow$ (ii) the key input is the "local criterion for flatness"

Important special cases:

① $\mathcal{F} = \mathcal{O}_X \rightsquigarrow X$ flat$/Y$?

② $X = Y \rightsquigarrow \mathcal{F}$ flat$/X$?

Local criterion for flatness (special case)

Let $A \to B$ be a local homom. of local noeth. rings, M a finitely generated B-module. Let $\mathfrak{a} \subsetneq A$ be a proper ideal.

The foll. are equivalent:

(i) M is flat $/A$

(ii) $M/\mathfrak{a}M$ is flat $/ A/\mathfrak{a}$ and the natural homom.

$$\mathrm{gr}^{\mathfrak{a}}(A) \otimes_{\mathrm{gr}^{\mathfrak{a}}_0(A)} \mathrm{gr}^{\mathfrak{a}}_0(M) \longrightarrow \mathrm{gr}^{\mathfrak{a}}(M) := \bigoplus \mathfrak{a}^n M / \mathfrak{a}^{n+1} M$$

is an isomorphism.

(iii) $M/\mathfrak{a}M$ is flat over $A/\mathfrak{a}$ and $\mathfrak{a} \otimes_A M \to M$ is injective.

Notation.

$t \in T \rightsquigarrow E_t := E \otimes_k k(t)$, the fiber of E_T over t

$x \in E(T) \rightsquigarrow (x, \mathrm{id}_T) : T \to E_T$,
$\underbrace{}_{}$ a closed immersion

$$\begin{tikzcd} T \arrow[r] \arrow[dr, "\mathrm{id}"'] & E_T \arrow[d] \\ & T \end{tikzcd} + E \text{ separated}$$

which defines an eff. Cartier divisor
(We need to check that the ideal sheaf $\mathcal{J}$ defining $T \hookrightarrow E_T$ is a line bundle, or equivalently, of finite presentation and flat (and thus locally free of finite rank; it is clear, e.g., by looking at fibers, that the rank must be $= 1$).

We would like to apply the fiber criterion for flatness ("special case ②" $X = Y := E_T$, $\mathcal{F} = \mathcal{J}$)

$$\begin{tikzcd} X = Y := E_T \arrow[d] \arrow[d, shift left] & \\ S := T & \end{tikzcd}$$

- $E_T \to T$ flat, $\mathcal{F}$ flat$/T$, $\mathcal{F}_t$ flat$/E_t$ $\forall t$
("Weil div = Cartier div." on the smooth curve E_t)
- need to check:
$\mathcal{J}$ of finite presentation,
but this is easy: locally on rings
$$\begin{tikzcd} A & A[X_\cdot]/(f_\cdot) \arrow[l, "\varphi"'] \\ & A \arrow[ul] \arrow[u] \end{tikzcd},$$
$\mathrm{Ker}(\varphi) = (X_i - a_i, i)$ finitely generated

Denote the corresponding line bundle by $\mathcal{O}_{E_T/T}(x)$ $\Big(:=$ the dual of the ideal sheaf defining $\mathrm{im}((x, id_T))\Big)$

Special case: $\mathcal{O}_{E_T/T}(0_T)$, $0_T : T \to \mathrm{Spec}\, k \xrightarrow{0} E$.

In view of the Yoneda lemma, it is then enough to show that

<u>Claim</u> For all T, the map (clearly functorial in T)

$$E(T) \longrightarrow \underline{\mathrm{Pic}}^0_{E/k}(T),$$
$$x \longmapsto \mathcal{O}_{E_T/T}(x) \otimes \mathcal{O}_{E_T/T}(0_T)^\vee,$$

is bijective.

Note that whenever T is the spec of a field, then the construction coincides with the bijection we discussed already.

<u>Proof of claim.</u> We construct a map in the other direction and show that the two maps are inverse to each other.

Given $\mathcal{L} \in \underline{\mathrm{Pic}}^0_{E/k}(T)$, write $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}_{E_T/T}(0_T)$, a line bundle on E_T which is fibrewise of degree 1. Denote the restriction to E_t by $\mathcal{L}'_t$.

Now for $\mathcal{L} \in \mathrm{Pic}^0(E_T)$.

Want: $\exists$ unique $x : T \to E$ $\exists$ $\mathcal{M} \in \mathrm{Pic}(T)$:

$$\mathcal{L} \cong \mathcal{O}_{E_T/T}(x) \otimes \mathcal{O}_{E_T/T}(0_T)^\vee \otimes p^*\mathcal{M},$$

IOW $\mathcal{L}' \cong \mathcal{O}_{E_T/T}(x) \otimes p^*\mathcal{M}$.

For $t \in T$, $\dim_{\kappa(t)} H^0(E_t, \mathcal{L}'_t) = \deg(\mathcal{L}'_t) = 1$,

$\dim_{\kappa(t)} H^1(E_t, \mathcal{L}'_t) = 0$ by RR/Serre duality

$\longrightarrow$ cohomology and base change for the proper morphism $E_T \to T$, [GW2] Cor 23.144, cf. AG3 Thm 4.3

$R^1 p_* \mathcal{L}' = 0$ (since $\beta^1(\kappa(t))$ surj $\forall t$)

$p_*\mathcal{L}'$ loc. free (of rank 1) (since β^1 surj. + β^0 surj.] $R^1 p_* \mathcal{L}' = 0$, so loc. free + β^{-1} surj.)

+ formation of $p_*\mathcal{L}'$ commutes with base change] β^0 surj.

May now replace $\mathcal{L}$ by $\mathcal{L} \otimes (p^* p_* \mathcal{L}')^\vee$

$\rightsquigarrow$ $p_*\mathcal{L}' \cong \mathcal{O}_T$ (by the projection formula).

Then $1 \in \Gamma(T, \mathcal{O}_T) = \Gamma(E_T, \mathcal{L}')$ defines an eff. Cartier divisor on E_T with assoc. line bundle $\mathcal{L}'$

$\rightsquigarrow$ closed subscheme $Z \subset E_T$. (cf AG2 Problem 33)

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This construction is compatible with base change, in particular $Z_t \subset E_t$ is eff. Cartier div. of degree 1 for all $t \in T$.

Then Z is flat over T by [AG1, Prop. 14.22], see above.

Also: Z finite $/T$

There are different ways of showing this:

- locally on T, we can find an embedding $E_T \hookrightarrow \mathbb{P}^N_T$ (in fact, we will see later that $N=2$ works)

which we can arrange so that $Z \cap V_+(X_2) = \emptyset$

$\rightsquigarrow$ Z is proper and affine $/T$ (and we may assume T affine)

Then use

<u>Lemma</u> Let $f: Z \to T$ be a proper and affine morphism of schemes. Then f is finite.

<u>Proof</u> (for T noetherian) WLOG T affine.

By coherence of (higher) direct images, $f_* \mathcal{O}_Z$ is a coherent $\mathcal{O}_T$-module, thus $\Gamma(Z, \mathcal{O}_Z)$ is a finite $\Gamma(T, \mathcal{O}_T)$-module.

General case: e.g., [Liu, Algebraic Geometry and Arithmetic Curves] Lemma 3.3.17.

• With (much) more effort, one can show the following result (a consequence of Zariski's Main Theorem):

<u>Theorem</u>. For a morphism $f: X \to Y$ the foll. are equiv:

(i) f finite

(ii) f is quasi-finite ($:=$ finite type and all fibers are finite sets)

and proper

(Reference: [GW 1] Cor 12.89.
For Y locally noetherian, see [Stacks] Ø20G for a different proof which relies on the theorem of formal functions.)

So the morphism $Z \to T$ is finite, flat and locally of finite pres.,

$\rightsquigarrow$ $Z \to T$ finite locally free of rank 1,

hence an isomorphism.

GW1, Cor. 12.19
or Stacks Ø2KB

<u>Lemma</u> Let $\varphi: R \to S$ be a ring homom so that S is a locally free R-module of rank 1 via φ. Then φ is an isom. (check that 1 is a basis of S).

$$\rightsquigarrow \quad \begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "\cong"'] & E_T \arrow[d] \\ & T \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} T \arrow[r, "\cong"] \arrow[rr, bend right, "x"'] & Z \arrow[r] & E_T \end{tikzcd}$$

This defines the desired map. It is not too hard to check that the two maps are inverse to each other.

$$= [\pm i] - 2[0]$$

$$\deg\left(\overbrace{\mathrm{div}\left(\frac{X^2+1}{X^2}\right)}\right)$$

$$= 0$$

Remark Degree of a divisor

For k a field,

C/k a curve,

$$D = \sum_{\substack{x \in C \\ \text{closed}}} n_x [x] \quad \text{a (Weil) divisor,}$$

we define $\quad \deg(D) = \sum_x n_x [\kappa(x) : k]$.

- If k is alg. closed, then $\kappa(x) = k$ for all $x \in C$ closed, so we can omit the terms $[\kappa(x) : k]$.

- If k is not alg. closed, it is important to include the degrees of the residue field extension. For example, this ensures that principal divisors on (proper) curves have degree 0.

Example: $k = \mathbb{R}$, $C = \mathbb{P}^1_{\mathbb{R}}$, $K(C) = \mathbb{R}(X)$

$$D = \mathrm{div}\left(\frac{X^2+1}{X^2}\right) = [\pm i] - 2[0].$$

Aside: The Picard functor

It is quite generally an interesting question whether, for a scheme X the set of (isom. classes of) line bundles can be equipped with a geometric structure

$\rightsquigarrow$ whether the Picard functor is representable.

Naive version: X an S-scheme

$$(Sch/S)^{op} \longrightarrow (Sets), \quad T \mapsto \mathrm{Pic}(X \times_S T)$$

Problem: not a sheaf for the Zariski topology!

There are tools to deal with this. For simplicity, we restrict to the following situation:

References: [GW2] Ch. 27 and the references given there,
[BLR, Néron models] Ch. 8

[GW2] (27.21), (27.22) and references given there

<u>Theorem</u> (representability of Picard functor)

Let $f: X \to S$ be a morphism.

Assume (a) $f_* \mathcal{O}_X = \mathcal{O}_S$ compatibly with base change

(b) f has a section

e.g. f flat, proper, of finite pres., with geom. reduced + geom. connected fibers

cf AG3 Pbm 46 / [GW2] Cor 24.63

Define

$$\underline{\mathrm{Pic}}_{X/S}(T) := \mathrm{Pic}(X \times_S T) / \mathrm{Pic}(T)$$

$\rightsquigarrow \underline{\mathrm{Pic}}_{X/S}: (\mathrm{Sch}/S)^{op} \to (\mathrm{Ab\,Grp})$

Can show: $\underline{\mathrm{Pic}}_{X/S}$ is a sheaf for the Zariski topology (cf AG3)

[and even for the "fppf topology"]

(1) If $S = \mathrm{Spec}\, k$, k a field, and f is proper, then $\underline{\mathrm{Pic}}_{X/k}$ is representable by a separated scheme lft/k.

(2) If f is flat, projective, of finite presentation and with geom. integral fibers, then $\underline{\mathrm{Pic}}_{X/S}$ is representable by a separated S-scheme locally of finite presentation.

The identity component of the Picard scheme

One can define the "identity component" $\underline{\mathrm{Pic}}^0_{X/S} \subset \mathrm{Pic}_{X/S}$.

In the situations of the representability theorems, this is a subgroup scheme s.t. for every $s \in S$,

$$(\mathrm{Pic}_{X_s/\kappa(s)})^0 = (\mathrm{Pic}_{X/S})^0 \times_S \mathrm{Spec}\,\kappa(s) \subset \mathrm{Pic}_{X/S} \times_S \mathrm{Spec}\,\kappa(s) = \mathrm{Pic}_{X_s/\kappa(s)}$$

is the identity component of the $\kappa(s)$-group scheme $\mathrm{Pic}_{X_s/\kappa(s)}$

[GW2] Section (27.24)

Important special cases:

- Let $C \to S$ be a smooth proper rel. curve with geometrically connected fibers [and admitting a section].

  Then $\underline{\mathrm{Pic}}^0_{C/S}$ is a smooth proper S-group scheme with geom. connected fibers of relative dim. g.

  (Say that $\underline{\mathrm{Pic}}^0_{C/S}$ is abelian scheme $/S$. For $S = \mathrm{Spec}$ of a field, it is an ab. variety.)

  We have $\underline{\mathrm{Pic}}^0_{C/S}(T) = \{ \mathcal{L} \in \mathrm{Pic}_{C/S}(T) ;$ $\mathcal{L}$ has degree 0 on each fiber of $C_T \to T \}$

The S-scheme $\underline{\mathrm{Pic}}^0_{C/S}$ is called the <u>Jacobian</u> of the relative curve C/S. This construction behaves functorially in C.

Abel-Jacobi map:

Assume there exists a line bundle $\mathcal{L}$ on C s.t. $\deg(\mathcal{L}_s) = 1$ for all s. (E.g. S the spectrum of a field k, $C(k) \neq \emptyset$)

Then morphism
$$C \longrightarrow \underline{\mathrm{Pic}}^0_{C/S}$$
$$x \longmapsto \mathcal{O}_C(x) \otimes \mathcal{L}^\vee$$

If all fibers C_s have genus >0, then this is a closed immersion.

The abelian variety "almost" determines the curve C, up to isomorphism (Theorem of Torelli, see [GW2] Rmk 27.287 for the statement and references).

- The dual abelian variety of an ab. variety; more generally: dual abelian scheme

For an abelian variety A over a field k,

$A^\vee = \underline{\mathrm{Pic}}^0_{A/k}$ is an abelian variety of dimension $\dim A$, called the <u>dual abelian variety</u> of A.

$\rightsquigarrow$ functor $(\mathrm{AbVar}/k)^{op} \to (\mathrm{AbVar}/k)$, $A \mapsto A^\vee$.

Similarly as for finite-dim'l v.s., one has natural morphism $A \to A^{\vee\vee}$ which is an isomorphism.

(Sometimes, but not in general, $A^\vee \cong A$.)

For $\mathcal{L}$ a line bundle on A, have $\varphi_{\mathcal{L}}: A \to A^\vee$

$$a \mapsto t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}.$$

(This is a group scheme homomorphism. Once we know that $A^\vee$ is an abelian variety, this follows since it clearly maps $0 \mapsto 0$. When this result is proved as a step towards the representability of the dual abelian variety, it is called the "Theorem of the square".)

Can show: (1) $\mathcal{L}$ ample $\implies$ $\varphi_{\mathcal{L}}$ surjective

(i.e. some power of $\mathcal{L}$ is $\cong \iota^* \mathcal{O}(1)$ for some embedding $A \overset{\iota}{\hookrightarrow} \mathbb{P}^N_k$)

(show that $\left(\mathcal{L} \otimes [-1]^* \mathcal{L}\right)|_{\ker(\varphi_{\mathcal{L}})}$ is at same time ample and trivial)

(2) $\mathcal{L} \in \mathrm{Pic}^0$ $\iff$ $\varphi_{\mathcal{L}} \equiv 0$, i.e. $\mathcal{L}$ is "translation invariant"

("$\implies$", admitting (1) + projectivity of A: choose $\mathcal{M}$ ample and write $\mathcal{L} = t_a^* \mathcal{M} \otimes \mathcal{M}^{-1}$. Then

$$t_b^* \mathcal{L} \otimes \mathcal{L}^{-1} = t_b^* (t_a^* \mathcal{M} \otimes \mathcal{M}^{-1}) \otimes (t_a^* \mathcal{M})^{-1} \otimes \mathcal{M}$$

$$= \underbrace{t_{a+b}^* \mathcal{M} \otimes \mathcal{M}^{-1}}_{\|\ \varphi_{\mathcal{M}}\text{ gp homom}} \otimes t_b^* \mathcal{M}^{-1} \otimes t_a^* \mathcal{M}^{-1} \otimes \mathcal{M}^2$$

$$t_a^* \mathcal{M} \otimes \mathcal{M}^{-1} \otimes t_b^* \mathcal{M} \otimes \mathcal{M}^{-1} \qquad)$$

Since $\underline{\mathrm{Pic}}^0$ is connected and contains the trivial line bundle $\mathcal{O}_A$, every $\mathcal{L} \in \underline{\mathrm{Pic}}^0(k)$ is algebraically equivalent to $\mathcal{O}$ (and since $\underline{\mathrm{Pic}}^0$ is the connected component of the trivial line bundle, $\underline{\mathrm{Pic}}^0(k)$ consists precisely of all line bundles on A alg. equiv. to $\mathcal{O}_A$).

For $\varphi: A \to A^\vee$, get $\Delta^*(\mathrm{id} \times \varphi)^* \mathcal{P}$, a line bundle on A, where $\mathcal{P}$ is the universal object of the representable functor $\underline{\mathrm{Pic}}^0_{A/k}$ ($\mathcal{P}$ is a line bundle on $A \times A^\vee$).

(This is not the inverse construction to $\mathcal{L} \mapsto \varphi_{\mathcal{L}}$.)

But after suitable modifications, one obtains close connection

$$\left\{ \begin{array}{c} \text{surjective homom.} \\ A \xrightarrow{\lambda} A^\vee \text{ s.t. } \lambda^\vee = \lambda \end{array} \right\} \longleftarrow \cdots \longrightarrow \left\{ \begin{array}{c} \text{embeddings} \\ A \hookrightarrow \mathbb{P}^N_k \end{array} \right\}$$