Example. If C/k is a curve which admits an embedding $C \hookrightarrow \mathbb{P}^2_k$ so that C is of degree 3, then by the genus-degree formula, C has genus 1.
(AG3, Problem 37)

(But note that $C(k)$ could be empty.)

Conversely, we have the following theorem.

Thm. Let E/k be an elliptic curve. Then there exist $a_i \in k$ s.t.

$$E \cong V_+(Y^2Z + a_1XYZ + a_3YZ^2 - X^3 - a_2X^2Z - a_4XZ^2 - a_6Z^3)$$
$$0 \longmapsto (0:1:0) \qquad \subseteq \mathbb{P}^2_k$$

("Weierstrass form" / "Weierstrass equation for E")

Proof. By Riemann-Roch, we have

$$\dim H^0(E, \mathcal{O}(n[0])) = n \text{ for all } n > 0.$$

We view $\mathcal{O}(n[0]) \subset \mathcal{K}_E$ ← constant sheaf attached to $K(E)$

as usual, in particular,

$$H^0(E, \mathcal{O}(n[0])) = \{ f \in K(E);\ \underbrace{\mathrm{div}(f) \geq -n[0]}_{\text{no poles on } E\setminus\{0\},\ \text{pole of order at most } n \text{ at } x} \}$$

⤳ then exist $x, y \in K(E)^\times$ s.t.

- $1, x$ basis of $H^0(E, \mathcal{O}(2[0]))$
- $1, x, y$ basis of $H^0(E, \mathcal{O}(3[0]))$

It follows that
- $1, x, y, x^2$ basis of $H^0(E, \mathcal{O}(4[0]))$,
- $1, x, y, x^2, xy$ basis of $H^0(E, \mathcal{O}(5[0]))$,
- $1, x, y, x^2, xy, x^3, y^2 \in H^0(E, \mathcal{O}(6[0]))$ linearly dependent

look at pole orders

⤳ $\exists a, b \in k^\times$, $a_i \in k$: $\quad a y^2 + a_1 xy + a_3 y = b x^3 + a_2 x^2 + a_4 x + a_6$

After change of coordinates $\left(\begin{matrix}\text{replace } x \text{ by } abx,\ y \text{ by } ab^2 y, \\ \text{divide equation by } a^3 b^4\end{matrix}\right)$,

may assume $a = b = 1$.

Write $F = Y^2 Z + a_1 XYZ + a_3 YZ^2 - (X^3 + a_2 X^2 Z + a_4 Z^2 + a_6 Z^3)$.

Since there is no $p \in E$ s.t. the sections $1, x, y$ of $\mathcal{O}(3[0])$ all vanish in the fiber $\mathcal{O}(3[0])(p)$

(... i.e. $\mathcal{O}(3[0])$ globally generated ...

- each of $1, x, y$ defines eff divisor lin.-equiv. to $3[0]$:

$$
\begin{array}{lcl}
1 & \longleftrightarrow & 3[0] \\
x & \longleftrightarrow & [0] + [?] + [?] \\
y & \longleftrightarrow & [?] + [?] + [?]
\end{array}
$$

$\rightsquigarrow$ 1 has zero of order 3 at 0

y has no zero at 0

OK)

we obtain a morphism $E \longrightarrow \mathbb{P}^2_k$

" $p \longmapsto (x(p) : y(p) : 1(p))$ "

(on $U = E \setminus \{0\}$, $\mathcal{O}(3[0])|_U \cong \mathcal{O}_U$

$\rightsquigarrow$ x, y define $U \to \mathbb{A}^1_k$

and the above morphism is given by $p \mapsto (x(p) : y(p) : 1)$;

for 0 we get, since y has higher pole order at 0 than 1 and x, $0 \longmapsto (0 : 1 : 0)$)

The morphism $E \to \mathbb{P}^2_k$ factors through the closed subscheme $V_t(F)$ by construction.

Claim The morphism $E \to V_t(F)$ is a closed immersion.

Admitting the claim, we get that $E \to V_t(F)$ is even an isomorphism because (as one checks by an elementary computation) F is irreducible, so that $V_t(F)$ is integral.

Then $E \to V_t(F)$ is a closed immersion of integral proper k-schemes of the same dimension, hence an isomorphism.

Rmk (1) If $\mathrm{char}(k) \neq 2, 3$, then by a suitable change of variables, the form of the eqn can be simplified to

$$Y^2 Z - X^3 - a X Z^2 - b Z^3 = 0$$

(2) Often one only writes the affine equation for E, e.g., $y^2 - x^3 - ax - b = 0$

(3) The point $(0:1:0)$ is the unique pt in $V_+(F) \cap V_+(Z)$.

(4) Given an eqn in Weierstraß form, the pt $(0:1:0)$ is always smooth, and the whole curve is smooth if and only

$$\Delta := \underbrace{-16(4a^3 + 27b^2)} \neq 0.$$

(We give the formula only for the simplified eqn. See [Tate, The Arithm. of Ell. C.] or [Silverman] or [GW2, Lemma 26.96] for the general case.

For $\mathrm{char}(k) \neq 2, 3$ (and the simplified form) this is equivalent to asking that the polynomial $X^3 + aX + b$ is seperable.

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Remark Let E/k be an ell. curve.

- Any two W-equations for E are related by a change of coordinates of the form

$$X = u^2 X' + r$$
$$Y = u^3 Y' + s u^2 X' + t \qquad u \in k^\times,\ r, s, t \in k.$$

(see e.g. [Silverman, Arithm. of Ell. C.] Ch III Prop. 3.1 (b)).

- For a change of coord. of the special form

$$X = u^2 X'$$
$$Y = u^3 Y'$$

we have $\quad a_i' = u^{-i} a_i.$

Still need to prove above 'claim': $E \to \mathbb{P}^2_k$ closed immersion.

Prop Let k be an alg. closed field and let $f: X \to Y$ be a proper morphism of k-schemes of finite type, such that

(a) f induces an injection $X(k) \hookrightarrow Y(k)$

(b) for all $x \in X$, f induces an injection $T_x X \hookrightarrow T_{f(x)} Y$ on the tangent spaces.

Then f is a closed immersion.

Rmk If f is a closed immersion, then (a) and (b) hold.

Proof

- (a) $\iff$ f injective
- Since f is proper + quasi-finite, it is finite.

  (Below we will apply the prop. to a morphism we know already to be finite.)
- Since f is injective and closed, it is a homeom. onto its image. It is therefore enough to show that the sheaf homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. We can check this on stalks where the result follows from the following lemma.

Lemma Let $\varphi : A \to B$ be a finite local homomorphism of noetherian local rings. Suppose that φ induces an isom. $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ of the residue class fields and a surjection $\mathfrak{m}_A/\mathfrak{m}_A^2 \twoheadrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ on the "cotangent spaces". Then φ is surjective.

Proof Since $\mathfrak{m}_A/\mathfrak{m}_A^2 \twoheadrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, we find that $\varphi(\mathfrak{m}_A)B = \mathfrak{m}_B$ by Nakayama's lemma $\left(\varphi(\mathfrak{m}_A)B + \mathfrak{m}_B^2 = \mathfrak{m}_B \Rightarrow \varphi(\mathfrak{m}_A)B = \mathfrak{m}_B\right)$.

Hence $B/\varphi(\mathfrak{m}_A)B = B/\mathfrak{m}_B = A/\mathfrak{m}_A$.

Applying Nakayama's lemma again, we see that B is generated as an A-module by the elt $1 = \varphi(1)$.

"Reminder" Ample line bundles.

[GW1] Sections (13.11), (13.12), (13.15)

[GW2] Section (23.2)

Let S be a scheme.

To simplify the situation a little, we assume that S is affine.

Let X be an S-scheme of finite type.

Def A line bundle $\mathcal{L}$ on X is called very ample (for $X \to S$), if there exist $n \geq 0$ and an immersion $X \overset{i}{\hookrightarrow} \mathbb{P}^n_S$ of S-schemes s.t.

$\mathcal{L} \cong i^* \mathcal{O}(1)$ (and then i is "given by" a family of $n+1$ global sections of $\mathcal{L}$ which generate $\mathcal{L}$).

Def A line bundle $\mathcal{L}$ on X is called <u>ample</u>, if the following equivalent conditions are satisfied:

(i) for every quasi-coh. $\mathcal{F}$-module of finite type, there exists $n_0 \in \mathbb{Z}$ s.t. for all $n \geq n_0$,

$$\mathcal{F} \otimes \mathcal{L}^{\otimes n} \text{ is generated by global sections}$$

(i') same for all qu-coh ideal sheaves $\mathcal{J} \subset \mathcal{O}_X$ of finite type

(ii) there exist $d \in \mathbb{Z}_{\geq 1}$ and finitely many sections $f_i \in \Gamma(X, \mathcal{L}^{\otimes d})$ s.t. all

$$X_{f_i} := \{x \in X;\ f_i(x) \neq 0\} \quad (\underset{\text{open}}{\subseteq} X)$$

are affine and $X = \bigcup_i X_{f_i}$.

(iii) [if S aff. + <u>noeth.</u> + X/S <u>proper</u>] for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there ex. $n_0 \geq 0$ s.t. $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ $\forall n \geq n_0, i > 0$

(iii') same for all coh. ideal sheaves $\mathcal{J} \subset \mathcal{O}_X$

(iv) there ex. $n > 0$ s.t. $\mathcal{L}^{\otimes n}$ is very ample.

<u>Remarks</u>. The "correct" def'n is (i). We will mostly use (iv).

We have proved (iv) $\Rightarrow$ (iii) in AG2. (iv) $\Rightarrow$ (ii) is easy.

Prop Let k be a field, k'/k a field extension,

X a k-scheme of finite type,

$\mathcal{L}$ a line bundle on X.

Then

$\mathcal{L}$ is (very) ample $\iff$ $\mathcal{L} \otimes_k k'$ is (very) ample

(for $X \to \operatorname{Spec} k$) (for $X \otimes_k k' \to \operatorname{Spec} k'$).

('$\Rightarrow$' is easy, for '$\Leftarrow$' see [GW1] Prop. 14.58).

Prop. Let k be an alg. cl. field, [GW2] Prop. 26.58

C a curve $/ k$,

D a divisor on C, $\mathcal{L} = \mathcal{O}_C(D)$.

(1) The foll. are equivalent:

(i) $\mathcal{L}$ is generated by global sections

(ii) $\forall x \in C$ closed:

$$\Gamma(C, \mathcal{O}(D - [x])) \subsetneq \Gamma(C, \mathcal{O}(D))$$

is a proper inclusion

(iii) $\forall x \in C$ closed:

$$\dim \Gamma(C, \mathcal{O}(D)) - \dim \Gamma(C, \mathcal{O}(D - [x])) = 1.$$

(2) The foll are equivalent:

(i) $\mathcal{L}$ very ample

(ii) $\forall x, y \in C$ closed:

$$\Gamma(C, \mathcal{O}(D - [x] - [y])) \subsetneq \Gamma(C, \mathcal{O}(D - [x])) \subsetneq \Gamma(C, \mathcal{O}(D))$$

are proper inclusions

(iii) $\forall x, y \in C$ closed:

$$\dim \Gamma(C, \mathcal{O}(D)) - \dim \Gamma(C, \mathcal{O}(D - [x] - [y])) = 2.$$

Proof (see [AWL] for details)

- (ii) $\Leftrightarrow$ (iii):

$$RR \Rightarrow \ 0 \leq \underbrace{\dim \Gamma(C, \mathcal{O}(D)) - \dim \Gamma(C, \mathcal{O}(D-[x]))}_{\parallel RR} \leq 1$$

from inclusion

$$\underbrace{\dim \Gamma(C, \mathcal{O}(-D+K_C)) - \dim \Gamma(\mathcal{O}(-D+[x]+K_C))}_{\leq 0} + 1$$

$$\mathcal{O}(-D+K_C) \subset \mathcal{O}(-D+[x]+K_C) \subset \mathcal{K}_C$$

- $\mathcal{L}$ glob. gen $\Longrightarrow \forall x \ \exists s \in \Gamma(C, \mathcal{L}) : \underbrace{s(x) \neq 0}$

$$\updownarrow$$

$$s \notin \Gamma(C, \mathcal{L} \otimes \mathcal{O}(-[x]))$$

- for (2), all cond. imply $\mathcal{L}$ glob. gen.

$\rightsquigarrow$ $\mathcal{L}$ defines $C \xrightarrow{f} \mathbb{P}(\Gamma(C, \mathcal{L}))$

- first incl. in (ii) for $x \neq y \iff f$ injective (on k-val. pts)

- — " — for $x = y \iff f$ inj. on tgt spaces (*)

(*) $x \in C$ closed $\rightsquigarrow T_x C \to T_{f(x)} \mathbb{P}^N_k$

inj. $\iff$ $\mathfrak{m}_{f(x)} / \mathfrak{m}_{f(x)}^2 \longrightarrow \underbrace{\mathfrak{m}_x / \mathfrak{m}_x^2}_{\text{1-dim'l } k\text{-v.s.}}$ surjective

$\iff$ —— " —— non-zero

$\iff$ $\exists\, s \in \Gamma(C, \mathcal{L})$,

$s \in \mathfrak{m}_x \mathcal{L}$, $s \notin \mathfrak{m}_x^2 \mathcal{L}$.

fix $\mathcal{O}(1)_{f(x)} \cong \mathcal{O}_{\mathbb{P}^1, f(x)} \rightsquigarrow \mathcal{L}_x \cong \mathcal{O}_{C,x}$ and comm. diag

$$\begin{tikzcd}
\mathfrak{m}_{f(x)}/\mathfrak{m}_{f(x)}^2 \arrow[r] & \mathfrak{m}_x/\mathfrak{m}_x^2 \\
\{ s \in \Gamma(\mathbb{P}^N, \mathcal{O}(1)),\ s(f(x)) = 0 \} \arrow[u, "\cong"] \arrow[r, two heads] & \{ s \in \Gamma(C, \mathcal{L});\ s(x) = 0 \} \arrow[u]
\end{tikzcd}$$

Cor. Let k be a field, and let C/k be a curve of genus g. Let $\mathcal{L}$ be a line bundle on C.

$\boxed{8.5.2024}$

(1) If $\deg(\mathcal{L}) \geq 2g$, then $\mathcal{L}$ is generated by global sections.

(2) If $\deg(\mathcal{L}) \geq 2g+1$, then $\mathcal{L}$ is very ample.

(3) $\mathcal{L}$ is ample if [and only if] $\deg(\mathcal{L}) > 0$.

Proof Use Riemann-Roch,

$$\dim H^0(C, \mathcal{L}) = 0 \text{ if } \deg(\mathcal{L}) < 0$$

$$\deg K_C = 2g-2$$

Alternative proof:

need to show: "="

$$E \longrightarrow V_+(F) \longrightarrow K(E) \hookleftarrow K(V_+(F)) = k(x,y)$$

nonconstant morph. of projective curves ⊛

Now $k(x) \hookrightarrow k(x,y) \hookrightarrow K(E)$

Corresponds to $E \to \mathbb{P}^1_k$ of degree 2 (fiber over ∞ is "$2\cdot[0]$")

$\longrightarrow$ $[K(E) : k(x,y)] \mid 2$

Similarly, from $k(y)$ we get $[K(E) : k(x,y)] \mid 3$.

⊛ At this point we do not know yet whether $V_+(F)$ is normal.
But if $V_+(F)$ is not normal, one shows (using that F is a polynomial in Weierstraß form) that ([Silverman] III. Prop 1.6) $K(V_+(F)) \cong K(\mathbb{P}^1_k)$ (equivalently: the normalization of $V_+(F)$ is $\cong \mathbb{P}^1_k$). But of course $K(E) \not\cong K(\mathbb{P}^1_k)$ since the field of rational functions determines the genus.
Therefore $V_+(F)$ is normal and $K(E) = K(V_+(F))$ implies $E \cong V_+(F)$.

Legendre family and j-invariant

In this section, $\boxed{\operatorname{char} k \neq 2}$

Let E/k be an elliptic curve,

and embed $E \cong V_+(Y^2 Z - (X^3 + a_2 X^2 Z + a_4 X Z^2 + a_6 Z^3)) \subset \mathbb{P}^2_k$

$\rightsquigarrow$ $E \cap \mathbb{A}^2_{k, x,y} \to \mathbb{P}^1_k$, $(x,y) \mapsto (x:1)$,

extends to a morphism $f: E \to \mathbb{P}^1_k$ of degree 2.

Lemma Assume that k is algebraically closed.

The morphism f is ramified over ∞ and over the (three distinct) zeros of $x^3 + a_2 x^2 + a_4 x + a_6$ in k.

Proof These are the points whose fiber contains only one point.

<u>Rmk</u> • The points in E where f is ramified are precisely the 2-torsion points of E,

i.e., writing $E[m] := \ker(E \xrightarrow{m} E)$, $m \in \mathbb{Z}$

($\subseteq E$ closed subgp scheme),

the 4 points in $E[2](k)$. [Problem 20 (1)]

• All other fibers (over closed points of $\mathbb{P}^1$) have the form $\{P, -P\}$ for some $P \in E(k)$,

so we can view $E \xrightarrow{f} \mathbb{P}^1$ as the "canonical projection to the quotient of E by the involution $P \mapsto -P$"

Proposition Let k be an algebraically closed field.

(1) Let E/k be an elliptic curve.

Then exist $\lambda \in \mathbb{P}^1(k) \setminus \{0, 1, \infty\}$ and a morphism $f : E \to \mathbb{P}^1_k$ which is ramified precisely at $0, 1, \lambda, \infty$.

For such λ, E is isomorphic to the curve E_λ given by (affine) W. eqn $y^2 - x(x-1)(x-\lambda)$.

(2) Let E, E' be ell. curves$/k$ and choose λ, λ' as in (1) for E, E', resp. Then

$$E \cong E' \iff \lambda' \in \left\{ \lambda, 1-\lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda} \right\} \quad (*)$$

Proof. (1) Given any three distinct points α, β, γ in $\mathbb{P}^1(k)$, there ex. a (unique) elt $\sigma \in PGL_2(k) = GL_2(k)/k^\times = \mathrm{Aut}_k(\mathbb{P}^1)$ s.t. $\sigma(\alpha) = 0$, $\sigma(\beta) = 1$, $\sigma(\gamma) = \infty$.

Now take $f: E \to \mathbb{P}^1$ as in (1), WLOG $f(0) = \infty$.

$\to$ $f \hat{=}$ basis of $H^0(E, \mathcal{O}(2[0]))$,

WLOG of the form $x, 1$

for some $x \in K(E)$ with a double pole at 0

[We know that $f(E \setminus \{0\}) \subseteq \mathbb{A}^1_k$.
Write the basis as $x, ax+b$,
$a, b \in k$. If $a \neq 0$, then $ax+b$ has
a zero on $E \setminus \{0\}$, say P, but
then $f(P) = (x(P) : 0) = (1:0) = \infty$,
a contradiction. So $a = 0$, and
by scaling we reduce to the case
$b = 1$.]

Continuing from this choice of x in the same way as before, we obtain a Weierstrass equation for E of the form $y^2 = g(x)$ where the zeros of g are the ramification pts $\neq \infty$ of the morphism f, i.e., $0, 1, \lambda$.

(2) "$\Rightarrow$" We may assume $E' = E$. Consider

$$\begin{array}{cc} E & E \\ \downarrow f \ \text{ramif over} & \downarrow f' \ \cdots 0, 1, \infty, \lambda' \\ \mathbb{P}^1 \quad 0, 1, \infty, \lambda & \mathbb{P}^1 \end{array}$$

WLOG $f(0) = \infty = f'(0)$ (compose with suitable translations)

$\rightsquigarrow$ $f^* \mathcal{O}(1) \cong \mathcal{O}_E(2[0]) \cong f'^* \mathcal{O}(1)$,

i.e. f, f' given by $\mathcal{O}_E(2[0])$

+ choice of basis of $H^0(E, \mathcal{O}_E(2[0]))$
(which could differ for f vs. f')

But change of basis $\longleftrightarrow$ autom. of $\mathbb{P}^1_k$ fixing ∞

$\updownarrow$

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL_2(k) ; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in PGL_2(k) \right\}$$

normalize so that $d = 1$

$0 \mathrel{\hat=} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} b \\ 1 \end{pmatrix}$

$1 \mathrel{\hat=} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a+b \\ 1 \end{pmatrix}$

$\lambda \mathrel{\hat=} \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a\lambda + b \\ 1 \end{pmatrix}$

$\rightsquigarrow$ $\{b, a+b, a\lambda + b\} = \{0, 1, \lambda'\}$

$\rightsquigarrow$ $\# S_3 = 6$ cases ...

e.g. $a+b = 0$, $a\lambda + b = 1$, $b = \lambda'$ $\rightsquigarrow$ $\lambda' = b = 1 - a\lambda = 1 + \lambda'\lambda$ $\rightsquigarrow$ $\lambda' = \frac{1}{1-\lambda}$

"$\Longleftarrow$" Replacing f, f' by the composition with a suitable automorphism of $\mathbb{P}^1$, we may assume that $f(0) = \infty = f'(0)$ and that $\lambda = \lambda'$.

But then $E' \cong E_\lambda \cong E$ by Part (1).

$\boxed{14.5.2024}$

<u>Corollary</u> $\#\,\mathrm{Aut}(E) < \infty$ (k-group scheme automorphisms)

(more precisely, $\#\,\mathrm{Aut}(E) \mid 12$ if char $k \neq 2$

$\#\,\mathrm{Aut}(E) \mid 24$ for all k).

(<u>Proof</u>: problem sheet)

The j-invariant of an elliptic curve

(we still assume $\mathrm{char}(k) \neq 2$)

Def Let k be an alg. closed field of char $\neq 2$, E/k an elliptic curve. Choose λ s.t.

$$E \cong \overline{V(y^2 - x(x-1)(x-\lambda))} = E_\lambda$$

Then
$$j(E) := 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} \in k$$

is called the j-invariant of E.

This is well defined since all possible choices for λ, given E, (see above) give rise to the same number.

(We can view $\lambda \mapsto j(\lambda)$ as a morphism $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 6, ramified over $\infty, 0, 1728$. (in char 3, $0 = 1728$ ...)

One can alternatively define $j(E)$ in terms of the coefficients of a Weierstraß equation for E, e.g. for $y^2 = x^3 + ax + b$, $\quad j = \frac{(4a)^3}{(4a^3+27b^2)}$

$\rightsquigarrow$ can define $j(E) \in k$ for E over arbitrary field k. (of arbitrary characteristic)

For k'/k, have $j(E \otimes_k k') = j(E)$.

Theorem Let k be a field, $\bar{k}$ an alg. closure of k. Let E, E' be ell. curves$/k$. Then

$$E \otimes_k \bar{k} \cong E' \otimes_k \bar{k} \iff j(E) = j(E').$$

Proof (for char $k \neq 2$)

"$\Rightarrow$" see above

"$\Leftarrow$" We may assume that $E = E_\lambda$,
$E' = E_{\lambda'}$

Since $\lambda \mapsto j(\lambda)$ has degree 6, one checks that $j(E') = j(E)$ implies that λ' is in the list $(*)$ (only need to check the ramification points), so $E \cong E_\lambda \cong E_{\lambda'} \cong E'$.

In particular, for k alg. closed:

$$\begin{aligned} \{\text{ell c.}/k\}/\cong \; &\longrightarrow k \\ E \; &\longmapsto j(E) \end{aligned}$$