Organizational stuff

$\boxed{9.4.2024}$

- schedule: lectures Tue 10
  Wed 12

  exercises Fri 14

https:// esaga / uni-due.de / ... / ss24 / alggeom4/

$\leadsto$ moodle page

Topic of course

elliptic curves ...
and their moduli spaces ...
as the Shimura variety for GL_2.

I. Shimura varieties

- rich topic, connects geometry (differential and algebraic), arithmetic, representation theory (of groups such as $GL_n(\mathbb{C})$, $GL_n(\mathbb{R})$, $GL_n(\mathbb{Q}_p)$, ...)

- the general definition is not so easy to digest:
  - (G, X) a Shimura datum
    - G connected reductive alg. group / $\mathbb{Q}$
    - X a $G(\mathbb{R})$-conj. class of homom $\mathbb{S} \to G_{\mathbb{R}}$,
      $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ "$= \mathbb{C}^\times$ as an alg. grp/$\mathbb{R}$"
      "Deligne torus"

Aside: The Deligne torus $\mathbb{S}$ (cf. Problem sheet 1)

- $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m$, i.e. $\mathbb{S}(T) = \mathbb{G}_m(T \otimes_{\mathbb{R}} \mathbb{C})$
$$= \Gamma(T_{\mathbb{C}}, \mathcal{O}_{T_{\mathbb{C}}})^\times,$$

in particular $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$

$\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$, $\quad \mathbb{S}_{\mathbb{C}} = \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$

cpl conj. τ

V a k-v.s., k a field

$$\rho : \mathbb{G}_m^r \to GL(V) \quad \overset{1-1}{\longleftrightarrow} \quad \text{grading } V = \bigoplus_{i \in \mathbb{Z}^r} V_i$$

[**Lemma** $\rho(z)$ diagonalizable (over k alg. cl.)] $V_i = \{ v \in V ; \rho(z) v = z^i v \ \forall z \in \mathbb{G}_m^r \}$

V an $\mathbb{R}$-v.s., $\rho : \mathbb{S} \to GL(V)$

$$\longleftrightarrow \quad V_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} V^{pq} \quad \circlearrowleft \text{ cpl conj.},$$

Hodge structure on V

$\circledast$ $\overline{V^{qp}} = V^{pq}$ inside $V_{\mathbb{C}}$.

$\left(\Longleftrightarrow \forall v \in V^{pq} \ \forall \underset{(\mathbb{C}^*)}{\hat{z}} : \rho(\bar{z}) \bar{v} = \bar{z}^{(q,p)} \bar{v} = \overline{z^{(p,q)} v} = \overline{\rho(z) v} \right)$

need to show:

$$\begin{array}{ccc} & \exists & \\ \mathbb{S} & \dashrightarrow & GL(V) \\ \uparrow & \llcorner & \uparrow \\ \mathbb{S}_{\mathbb{C}} & \xrightarrow[\varphi]{} & GL(V_{\mathbb{C}}) \end{array} \quad \Longleftrightarrow \quad \circledast$$

descent theory, have $\Gamma(\mathbb{S}, \mathcal{O}_{\mathbb{S}}) = \Gamma(\mathbb{S}_{\mathbb{C}}, \mathcal{O})^{\tau}$, same for GL_n

$\Updownarrow$

φ comp. with cpl conj.

such that

(SV1) $\forall h \in X$, the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ def'd by $\mathrm{Ad} \circ h : \mathbb{S} \to G_{\mathbb{R}} \to GL(\mathrm{Lie}(G_{\mathbb{R}}))$ is of type $\{(-1,1), (0,0), (1,-1)\}$

(SV2) $\forall h \in X$: $\mathrm{Int}(h(i))$ is a Cartan invol. of $G^{ad}_{\mathbb{R}}$

(SV3) G^{ad} has no $\mathbb{Q}$-factor on which h is trivial

$G^{ad} = G/Z(G)$, $Z(G) \subseteq G$ the center of G (the "adjoint group" of G)

– $K \subset G(\mathbb{A}_f)$ compact open, [usually assume sufficiently small]

($\mathbb{A}_f = \left(\prod_p \mathbb{Z}_p\right) \otimes_{\mathbb{Z}} \mathbb{Q}$ finite adeles, topological ring)

$$\rightsquigarrow \quad S_K(\mathbb{C}) = {}_{G(\mathbb{Q})}\backslash \left(X \times G(\mathbb{A}_f)/K \right),$$

<u>Thm</u> $S_K(\mathbb{C})$ = the $\mathbb{C}$-valued pts of a quasi-proj. [smooth] complex alg variety S_K.

<u>Example</u>

(1) GL_2, $h_0(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$,

$X \cong \mathbb{C} \setminus \mathbb{R}$, $h_0 \leftrightarrow i$ $\quad$ ($GL_2(\mathbb{R}) \curvearrowright \mathbb{C}\setminus\mathbb{R}$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$)

(2) $\nexists$ Shimura datum (G,X) for $G = GL_n$, $n > 2$.

Cartan involution. $G/\mathbb{R}$ linear alg. gp

$\vartheta: G \to G$ involution (gp. autom with $\vartheta^2 = id$)

s.t. $G^{(\vartheta)} = \{ g ; \vartheta(g) = \bar{g} \}$ compact.

(exists iff G reductive iff G has cpt real form)

Example: $G = GL_n$, $g \mapsto (g^t)^{-1}$.

If G has a Cartan invol ϑ, then can embed $G \hookrightarrow GL_n$ s.t. ϑ is the restriction of $g \mapsto (g^t)^{-1}$ to G.

Example For $n > 2$, there is no Shimura datum for GL_n.

In fact, the (basically unique) Cartan involution is $g \mapsto (g^t)^{-1}$, but in general $(g^t)^{-1}$ is not conjugate to g in PGL_n (not even for g a diagonal matrix).

(compare GL_2: $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} a_2 & \\ & a_1 \end{pmatrix} = \det \cdot \begin{pmatrix} a_1^{-1} & \\ & a_2^{-1} \end{pmatrix}$)

Structure of $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$:

$$\begin{array}{ccc} G(\mathbb{R})_+ & \hookrightarrow & G(\mathbb{R}) \\ \downarrow & \square & \downarrow \\ G^{ad}(\mathbb{R})^+ & \subseteq & G^{ad}(\mathbb{R}) \end{array}$$

identity comp.

- $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$ with

  Let $X^+ \subseteq X$ be a connected comp.

  Then $G(\mathbb{Q})_+ \backslash (X^+ \times G(\mathbb{A}_f)) \longrightarrow G(\mathbb{Q})\backslash (X \times G(\mathbb{A}_f))$

  is a bijection. (Key pt: $G(\mathbb{Q}) \subseteq G(\mathbb{R})$ dense "real approxim.")

- $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$ is a <u>finite</u> set

Keypts: ① Strong approximation: $G/\mathbb{Q}$ semisimple, simply connected, of non-cpt type

Examples: SL_n, Sp_{2n}

cpt type: $G(\mathbb{R})$ cpt

non cpt type: does not contain non-trivial normal subgp of cpt type

$\Rightarrow$ $G(\mathbb{Q}) \subset G(\mathbb{A}_f)$ dense

[OTOH, $G(\mathbb{Q}) \subset G(\mathbb{A}_f)$ not dense for, e.g., $G = \mathbb{G}_m$, $G = PGL_n$ Milne-SVI Rmk 4.17]

From strong approx: $\rightsquigarrow$ $G(\mathbb{Q}) \underset{\text{dense}}{\subset} \underbrace{G(\mathbb{A}_f)/K}_{\text{discrete}}$

$K \subset G(\mathbb{A}_f)$ open

$\rightsquigarrow$ $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K = \{pt\}$

With a bit more work, this reduces the question to the case of tori.

② $T/\mathbb{Q}$ torus $\rightsquigarrow$ $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/T(\hat{\mathbb{Z}})$

finite, where

$T(\mathbb{Z}_\ell) = \{a \in T(\mathbb{Q}_\ell);\ \chi(a) \in \mathbb{Z}_\ell^\times\ \forall \chi \in X^*(T)\}$

$T(\hat{\mathbb{Z}}) = \prod T(\mathbb{Z}_\ell)$

Example

If $F/\mathbb{Q}$ finite, $T = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ "$= F^\times$ as alg. gp$/\mathbb{Q}$",

then $T_{\bar{\mathbb{Q}}} = \prod_{\sigma : F \hookrightarrow \bar{\mathbb{Q}}} \mathbb{G}_{m,\bar{\mathbb{Q}}}$, $X^*(T) = \mathbb{Z}^{\mathrm{Hom}(F,\bar{\mathbb{Q}})}$,

$$T(\mathbb{Z}_\ell) = (\mathbb{Z}_\ell \otimes_{\mathbb{Q}} F)^\times = \prod_{\substack{v \in Pl(F) \\ v \mid \ell}} \mathcal{O}_{F,v}^\times, \quad T(\hat{\mathbb{Z}}) = \prod_v \mathcal{O}_{F,v}^\times,$$

$$\rightsquigarrow T(\mathbb{Q})\backslash T(\mathbb{A}_f)/T(\hat{\mathbb{Z}}) = \underbrace{F^\times}_{\text{principal divisors}} \backslash \underbrace{\mathbb{A}_{F,f}^\times / \prod \mathcal{O}_{F,v}^\times}_{= Z^1(\mathrm{Spec}\,\mathcal{O}_F) = \text{the gp of Weil divisors on } \mathrm{Spec}\,\mathcal{O}_F}$$

$= Cl(F)$, the class group of F.

So in this case, the statement becomes the theorem of the finiteness of the class group.

- Now let $\mathcal{C} \subset G(\mathbb{A}_f)$ be a system of represent. for $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$.

For $g \in \mathcal{C}$, put $\Gamma_g = gKg^{-1} \cap G(\mathbb{Q})_+$.

Then there is a natural homeomorphism

$$G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/K) \xleftrightarrow{1:1} \coprod_{g \in \mathcal{C}} \underbrace{\Gamma_g \backslash X^+}_{\ni} .$$

$$[x, g] \longmapsto x$$

$\boxed{10.4.2024}$

<u>Rmk</u> X^+ (equivalently, X) carries a natural complex structure: Write $\mathfrak{g} = \mathrm{Lie}(G_{\mathbb{R}})$

- $T_h X^+$ (the tgt space of the real mf X^+ at $h \in X^+$) can be identified with $\mathfrak{g}/L^{00}$ (since $L^{00} = \mathrm{Lie}(\mathrm{Stab}_{G_{\mathbb{R}}}(h))$) where $L^{00} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_{\mathbb{C}}^{00} \subseteq \mathfrak{g}_{\mathbb{C}}$

and

$$\mathfrak{g}/L^{00} \xrightarrow{\sim} \mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}},$$

$$F^r\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p \geq r} \mathfrak{g}_{\mathbb{C}}^{p,q}$$

a <u>complex</u> vector space.

- Similarly, given any injective representation $G_{\mathbb{R}} \hookrightarrow GL(V)$, the map

$$X^+ \xrightarrow{\varphi} \mathrm{Flag}(V_{\mathbb{C}})$$
$$h \longmapsto (F^r)_r$$

the Hodge filtration given by h

has the property that $(d\varphi)_h$ identifies $T_h X^+$ with a complex subvector space of $T_{\varphi(h)}\,\mathrm{Flag}$. It follows that φ identifies X^+ with an open (complex!) submanifold of Flag.

Until this point we used only part of axiom (SV 1): it was enough to know that each h induces, on $\mathrm{Lie}(G_{\mathbb{R}})$, a Hodge structure of weight 0, i.e. $p+q=0$ $\forall\, p,q$ with $V^{pq} \neq 0$.

Moreover, one can show: ([Deligne, Corvallis, 1.1.14])

- (SV 1) $\iff$ for $G_{\mathbb{R}} \to GL(V)$, the family of HS on V given by X is a "variation of Hodge structures", i.e. satisfies Griffiths transversality

(1.5 lines in loc. cit.)

- (SV 2) $\iff$ for $G_{\mathbb{R}} \to GL(V)$ defining a family of HS on V of weight n, there ex. $\psi : V \otimes V \to \mathbb{R}(-n)$ which induces for each h a polarization of the HS given by h.

From "complex" to "algebraic"

Thm (Baily-Borel) The complex space $\Gamma_g \backslash X^+$ is (the analytification of) a quasi-projective algebraic variety $/\mathbb{C}$ (smooth if Γ suff. small), i.e. an open subvariety of a projective (normal, but typically non-smooth) $\mathbb{C}$-scheme $(\Gamma_g \backslash X^+)^*$, the BB compactification of $\Gamma \backslash X^+$.

Remark For simplicity, we phrased the BB thm for the quotients $\Gamma \backslash X^+$ arising from a Shimura datum as explained above.
What one needs is that
- X^+ is a hermitian symmetric domain
- $\Gamma \subset \mathrm{Hol}(X^+)^+$ is an <u>arithmetic</u> subgroup (a crucial condition for getting algebraicity and a severe restriction on Γ.

(For an alg. gp $G/\mathbb{Q}$, a subgp. $\Gamma \subset G(\mathbb{Q})$ is arithmetic, if for some (equiv. for every) algebraic embedding $G \hookrightarrow GL_n(\mathbb{Q})$ as a closed subgroup, $\Gamma \cap GL_n(\mathbb{Z})$ has finite index in Γ and in $GL_n(\mathbb{Z})$ ("Γ is <u>commensurable</u> with $GL_n(\mathbb{Z})$"))

A subgroup Γ of a connected real Lie group H (e.g., $H = \mathrm{Hol}(X^+)^+$, the identity component of the group of automorphisms of the complex manifold X^+), is arithmetic, if there exist an alg. group $G/\mathbb{Q}$, an arithmetic subgroup $\Gamma_0 \subset G(\mathbb{Q})$, a surjective homomorphism

$G(\mathbb{R})^+ \to H$ of real Lie groups with compact kernel s.t. $\Gamma_0 \cap G(\mathbb{R})^+$ maps onto Γ.

For a Shimura datum (G, X) and $K \subset G(\mathbb{A}_f)$ compact open, $K \cap G(\mathbb{Q}) \subset G(\mathbb{Q})$ is a "congruence subgroup", i.e. contains $\Gamma(N) = G(\mathbb{Q}) \cap \Gamma_{GL_n}(N)$ for some embedding $G \subseteq GL_n$. A fortiori, $K \cap G(\mathbb{Q})$ is arithmetic.

Conversely, every congruence subgroup of $G(\mathbb{Q})$ is of the form $K \cap G(\mathbb{Q})$ for some $K \subset G(\mathbb{A}_f)$ compact open. In particular, the notion of congruence subgroup is independent of the embedding

__Thm__ (Borel) The algebraic structure on $\Gamma_g \backslash X^+$ is uniquely determined.

__Reference__ [Milne, Intro to Shimura var's], Ch. 3, Ch. 5.

Aside The analytification functor ([GW2] Section (20.12))

$$\left(\text{Sch lft}/\mathbb{C}\right) \to \begin{pmatrix}\text{complex}\\ \text{analytic}\\ \text{spaces}\end{pmatrix} \quad \text{s.t.} \quad X^{an} \xleftrightarrow{1:1} X(\mathbb{C})$$

$$X \longmapsto X^{an}$$

locally ringed spaces that look locally like zero sets of finitely many holomorphic fcts on $\mathbb{C}^n$ (locally for analytic topology, structure sheaf on $\mathbb{C}^n$ given by holomorphic fcts)

Dictionary for properties of X / X^{an}:

X	X^{an}
smooth/$\mathbb{C}$	complex manifold
separated	Hausdorff
proper	Compact (:= Hausdorff + quasi-compact)
isomorphism	isomorphism

<u>NB</u> • $X^{an} \cong Y^{an} \not\Rightarrow X \cong Y$

• $\_^{an}$ is not a full functor: $\exp : \mathbb{C} \to \mathbb{C}$ $\quad (\mathbb{C} = \mathbb{A}^{1,an}_{\mathbb{C}})$

<u>Theorem</u> (1) $\_^{an} : \begin{pmatrix}\text{proper}\\ \mathbb{C}\text{-sch.}\end{pmatrix} \to \begin{pmatrix}\text{cpt cpl}\\ \text{anal sp.}\end{pmatrix}$ fully faithful

(2) <u>Chow's theorem</u>

$$\_^{an} : \begin{pmatrix}\text{projective}\\ \mathbb{C}\text{-schemes}\end{pmatrix} \to \begin{pmatrix}\text{cpl anal sp. which}\\ \text{admit closed anal. emb.} \hookrightarrow \mathbb{P}^n(\mathbb{C})\end{pmatrix} \text{ equiv. of cat.}$$

(use GAGA ... then for (1) represent morphisms by their graphs)

We will discuss the theorem of Baily and Borel in the case of modular curves in more detail below.

The theorem of Borel rests on the following surprising fact

<u>Thm</u> Consider $\Gamma\backslash X^+$ as before, and assume Γ is torsion-free. Let V be a smooth quasi-proj. $\mathbb{C}$-scheme. Then every holomorphic map $V^{an} \to \Gamma\backslash X^+$ is the analytification of a morphism of $\mathbb{C}$-schemes.

<u>Key lemma</u> Let D be the open cpl. unit disk, $D = \{z \in \mathbb{C};\ |z|<1\}$, and $D^* = D \setminus \{0\}$ be the punctured disk.
Every holom. map $(D^\times)^r \times D^s \to \Gamma\backslash X^+$ extends to a holom. map $D^{r+s} \to (\Gamma\backslash X^+)^*$ of anl. spaces.

[cf. Picard's big theorem: If f is holom. on $D^\times$ with an essential singularity at 0 then $\# \mathbb{C} \setminus f(D^\times) \leq 1$.
IOW: every holomorphic fct $D^\times \to \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$ extends to $D \to \mathbb{P}^1(\mathbb{C})$. $\rfloor$

Given the lemma, to prove the theorem let

V^* proj. smooth $/\mathbb{C}$ s.t. $V \subseteq V^*$ open dense and

$V^* \setminus V$ is a divisor with normal crossings (dnc)

(the existence of V^* follows from Hironaka's theorem on resolution of singularities).

Then locally the inclusion $V \hookrightarrow V^*$ looks like

$(D^\times)^r \times D^s \hookrightarrow D^{r+s}$. By the lemma we obtain

$(V^*)^{an} \longrightarrow (\Gamma\backslash X^+)^*$ and the result follows from

Chow's theorem.

proof

$\leadsto$ the BB compactification has the following minimality property:

$$\left[\begin{tikzcd} & V^* \text{ smooth cpt} \arrow[d, dashed] \\ \Gamma\backslash X^+ \arrow[ur, "\text{open, dense, bdry a dnc}"] \arrow[r, phantom, "\subset"] & (\Gamma\backslash X^+)^* \end{tikzcd}\right.$$

(and is therefore sometimes called the minimal compactification

References Milne - SVI, Ch. 3

The $G(\mathbb{A}_f)$-action

16.4.2024

For $K' \subseteq K \subset G(\mathbb{A}_f)$ cpt open, suff. small, have projection $S_{K'} \to S_K$, a finite étale morphism.

$\rightsquigarrow$ inverse system $(S_K)_K$

For $h \in G(\mathbb{A}_f)$, have isomorphism

$$\begin{aligned} {}_{G(\mathbb{Q})}\backslash\left(X \times G(\mathbb{A}_f)/_{K}\right) &\xrightarrow{\sim} {}_{G(\mathbb{Q})}\backslash\left(X \times G(\mathbb{A}_f)/_{h^{-1}Kh}\right) \\ (x, g) &\longmapsto (x, gh) \end{aligned}$$

gh

$\rightsquigarrow$ "$G(\mathbb{A}_f)$ acts on the inverse system $(S_K)_K$"

(One can also pass to the inverse limit scheme $\lim S_K$ which acquires an honest $G(\mathbb{A}_f)$-action (but is not of finite type $/\mathbb{C}$ anymore).)

Theorem (Shimura, Deligne)

The variety S_K is defined over a number field E (the "reflex field") (!) which is independent of K (!), more precisely there ex. a "canonical" model $\underline{S}_K / E$ s.t. $\underline{S}_K \otimes_E \mathbb{C} \cong S_K$, and the $G(\mathbb{A}_f)$-action 'descends' to $(\underline{S}_K)_K$

Foundational idea for "canonical" (think of this as a uniqueness statement):

Lemma Let K be a field of char. 0 and let L be an alg. closed extension field of K. The natural functor

$$\begin{pmatrix}\text{reduced separated} \\ \text{finite type schemes}/K\end{pmatrix} \longrightarrow \begin{pmatrix}\text{reduced sep.} \\ \text{ft schemes } Y/L \\ + \operatorname{Aut}(L/K) \curvearrowright Y(L)\end{pmatrix}$$

$$X \longmapsto X \otimes_K L + \underbrace{\operatorname{Aut}(L/K) \curvearrowright X(L)}_{\text{natural action}}$$

is fully faithful.

$\rightsquigarrow$ need to pin down the action of $\mathrm{Aut}(\mathbb{C}/E)$ on $S_K(\mathbb{C})$ ...

IOW: assume M_1, M_2 are canonical models of S_K $\rightsquigarrow$ get two (possibly different) actions $\mathrm{Aut}(\mathbb{C}/E) \circlearrowright S_K(\mathbb{C})$, and we need to show they coincide.

$\rightsquigarrow$ enough to find a generating set of $\mathrm{Aut}(\mathbb{C}/E)$ whose elements each coincide on a Zariski dense subset of $S_K(\mathbb{C})$

Consequence: $\quad \mathrm{Gal}(\bar{E}/E) \circlearrowright H^i(\underline{S}_K \otimes_E \bar{E}, \bar{\mathbb{Q}}_\ell)$

also $\quad G(\mathbb{A}_f) \circlearrowright \operatorname{colim}_K H^i(\underline{S}_K \otimes_E \bar{E}, \bar{\mathbb{Q}}_\ell)$

$K' \subset K$

$\rightsquigarrow S_{K'} \to S_K$

$\rightsquigarrow H^i(S_K) \to H^i(S_{K'})$

These actions commute with each other since the $G(\mathbb{A}_f)$-action is defined over E.

$\rightsquigarrow$ S_K is a tool to compare repns. of $\mathrm{Gal}(\bar{E}/E)$ and of $G(\mathbb{A}_f)$. $\longleftrightarrow$ Langlands program

(This is overly simplified... In particular, $\underline{S}_K$ typically is not proper ($\longleftrightarrow$ $S_K(\mathbb{C})$ not compact), and then $H^i(\underline{S}_K \otimes \bar{E}, \bar{\mathbb{Q}}_\ell)$ is not the right object to work with.)