ALGEBRAIC GEOMETRY 3, WINTER TERM 2023/24. LECTURE COURSE NOTES.

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1. Introduction

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These notes are not complete lecture notes, but should rather be thought of as a rough summary of the content of the course. Many proofs are only sketched, or are omitted. Please do not hesitate to ask for details whenever the given information is not sufficient!

References. The books [GW1], [GW2] by Wedhorn and myself, by Hartshorne [H], and the Stacks project [Stacks]. The book by Vakil [Va] is also recommended. More precise references are given in most of the individual sections.

This lecture course is a continuation of the courses *Algebraic Geometry 1*, *Algebraic Geometry 2* which covered the definition of schemes, some basic notions about schemes and scheme morphisms: Reduced and integral schemes, immersions and subschemes, and fiber products of schemes, separated and proper morphisms, $\mathscr{O}_X$-modules, line bundles and divisors, and basics of the cohomology of $\mathscr{O}_X$-modules including the standard vanishing theorems (Grothendieck vanishing, vanishing of higher cohomology of affine schemes with coefficients in quasi-coherent sheaves) and the finiteness of cohomology for projective schemes.

Outline of this course:

- Smoothness and differentials – The notion of smoothness is very important throughout algebraic geometry, so it is high time that we cover it in the lectures. Furthermore, it is closely related to the notion of differential forms (of course, we need a suitable algebraic form of this). As it will turn out, sheaves of differential forms are in turn closely related to Serre duality, a topic that we have already scratched and that we will come back to in this class.
- Serre duality – We have seen in the last term that Serre duality, while the statement itself is a bit technical, has nice consequences such as the Theorem of Riemann-Roch. With the theory of differentials at hand, it will not be so difficult to develop the cohomological machinery until the point where we can prove it for a large class of schemes.
- Cohomology and base change – Another crucial technique to study cohomology goes under the name of cohomology and base changes.

It concerns the following question: Given a morphism $f\colon X \to Y$ of schemes, and a quasi-coherent $\mathscr{O}_X$-modules $\mathscr{F}$, under which conditions is the natural $\kappa(y)$-vector space homomorphism

$$R^i f_* \mathscr{F} \otimes \kappa(y) \to H^i(X_y, \mathscr{F}_{|X_y})$$

an isomorphism? Here $y \in Y$ and $X_y := X \times_Y \operatorname{Spec} \kappa(y)$ is the scheme-theoretic fiber of f over y.

- Grassmannians, flag varieties, Schubert varieties – These are classes of varieties that have a rather explicit definition, and on the other hand are quite interesting and occur in many different contexts. Grassmannians are natural generalizations of projective space; they parameterize r-dimensional subspaces of an n-dimensional vector space. (I.e., for $r = 1$ we obtain the projective space $\mathbb{P}^{n-1}$.)
- Hilbert schemes – If there is time, I will discuss the construction of the Hilbert scheme. Similarly as for projective space and for Grassmannians, it is (relatively) easy to write down the functor of T-valued points of the Hilbert scheme. However, in this case it is far from obvious that a scheme giving rise to this functor exists. A crucial ingredient of the proof is the cohomologocal machinery we have developed, in particular the theory of cohomology and base change.

I want to give some pointers to results that answer natural questions that can be answered by the above tools (in particular, the machinery of cohomology), and which can be stated without any reference to this.

Remark: Quasi-projectivity of curves 1.1. Let k be a field. In this remark, by a *curve* over k we mean a separated finite type k-scheme of dimension 1, i.e., every local ring of a closed point has dimension 1.

Theorem. Let C be a curve. Then C is a quasi-projective k-scheme, i.e., C is isomorphic to a locally closed subscheme of some projective space.

We will not discuss the complete proof of the theorem here, but only sketch some of the steps.

(I) Assume that C is normal, i.e., all local rings of C at closed points are discrete valuation rings. The key point is then that the projective space $\mathbb{P}^n_k$ satisfies the *valuative criterion of properness*:

Theorem 1.2. (Valuative criterion of properness, noetherian version, [GW1] Theorem 15.9)

Let S be a noetherian scheme and let $f\colon X \to Y$ be a morphism of finite type. We consider commutative diagrams of the form

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y, \end{array}$$

where R is a discrete valuation ring with field of fractions K and the vertical arrow on the left is the canonical inclusion.

The following are equivalent:

(i) *The morphism f is proper (resp., separated).*

(ii) *In every diagram as above there exists a unique (resp., at most one) morphism* $\operatorname{Spec} R \to X$ *making the resulting diagram commutative.*

We have proved in AG2 that the structure morphism $\mathbb{P}^n_k \to \operatorname{Spec} k$ is proper. It follows that it has the property stated in the criterion. However, since we did not prove the valuative criterion of properness, in class we verified directly that it holds for $\mathbb{P}^n_k$ over k. This is easy using the description of S-valued points of projective space; for S the spectrum of a local ring, one basically obtains a description in terms of homogeneous coordinates (cf. [GW1] Exer. 4.6 for a precise statement). Proving (ii) $\Rightarrow$ (ii) in the valuative criterion would now give a different proof of the fact that projective space is proper over the base.

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Now let $U \subseteq C$ be open affine and choose an immersion $f\colon U \hookrightarrow \mathbb{A}^n_k \hookrightarrow \mathbb{P}^n_k$. For $x \in C \setminus U$, the above shows that we can extend the morphism $\operatorname{Spec} K(C) \to \mathbb{P}^n_k$ given by f to a morphism $\operatorname{Spec} \mathscr{O}_{C,x} \to \mathbb{P}^n_k$. This morphism can be extended to some open neighborhood V of x (view $\mathscr{O}_{C,x}$ as the localization of $\Gamma(V, \mathscr{O}_C)$ with respect to some prime ideal; the image of $\operatorname{Spec} \mathscr{O}_{C,x}$ is contained in one of the standard open charts of $\mathbb{P}^n_k$, which we can write as $k[X_1, \dots, X_n]$; in the images of the X_i in $\mathscr{O}_{C,x}$ only finitely many denominators are involved, hence on the ring level the homomorphism factors through the localization with respect to a suitable element s; for the scheme morphism this means that it extends to $D(s) \subseteq V$; see [GW1] Prop. 10.52). Since C is reduced and $\mathbb{P}^n_k$ is separated, and the morphisms $U \to \mathbb{P}^n_k$ and $V \to \mathbb{P}^n_k$ coincide on $\operatorname{Spec} K(C)$, which is dense, they actually coincide on $U \cap V$ (AG2, Problem [GW1] Cor. 9.9) and we can glue them. Repeating this, if necessary, we can extend the morphism $U \to \mathbb{P}^n_k$ to a morphism $C \to \mathbb{P}^n_k$.

At this point, the choice of an affine open U and an immersion $U \to \mathbb{P}^n_k$ gives us a morphism $C \to \mathbb{P}^n_k$. However, in general this will not be an immersion. To finish the proof of the theorem for normal curves, we proceed as follows. Let $C = \bigcup_{i=1}^m U - i$ be an affine open cover. For each U_i, as above we obtain a morphism $f_i\colon C \to \mathbb{P}^{n_i}_k$ such that $f_{i|U_i}$ is an immersion. This gives us a morphism $C \to \prod_i \mathbb{P}^{n_i}_k$ 8where the product is the fiber product over $\operatorname{Spec} k$), and composing this morphism with the Segre embedding $\prod_i \mathbb{P}^{n_i}_k \to \mathbb{P}^N_k$ (where N depends on the n_i as dictated by the Segre embedding, a closed embedding; see [GW1] Section (4.14)) we obtain a morphism $f\colon C \to \mathbb{P}^N_k$ such that for every i, the restriction $f_{|U_i}$ is an immersion. We can then conclude by the following lemma.

Lemma 1.3. ([GW1] Lemma 14.18) *Let X be a scheme which has only finitely many irreducible components. Let $f\colon X \to Y$ be a separated morphism. Assume that there exists a cover $X = \bigcup_i U_i$ where each U_i is open* *and dense in X, and such that for every i the restriction $f_{|U_i}$ is an (open) immersion. Then f is an (open) immersion.*

Proof. The issue here is to show that under the given assumptions, f is injective.

We sketch the proof in case X is irreducible and all $f_{|U_i}$ are open immersions, which is the case relevant for us. By replacing Y with the reduced closure of the image of f, we reduce to the case that Y is integral and that f is dominant (i.e., the image of f is dense in Y). It follows that for every $x \in X$, the ring homomorphism $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is an isomorphism and thus in particular *flat*.

That all these ring homomorphisms are flat is usually expressed by saying that f is flat. This property is stable under base change. We will use that whenever $\varphi\colon A \to B$ is a flat local ring homomorphism of local rings, then the induced morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective; this is a commutative algebra result which sometimes goes by the name *going down for flat morphisms*, see e.g. [GW1] Example B.18, [M2] Theorem 9.5. A *maximal point* of a scheme is a point x such that there exists no point $x' \neq x$ with $x \in \overline{\{x'\}}$, i.e., x is a generic point of an irreducible component. As a consequence of the above discussion we see that under a flat morphism $X \to Y$ of schemes, every maximal point of X is mapped to a maximal point of Y:

Coming back to the specific situation at hand, to show that f is injective, we will show that the diagonal morphism $\Delta\colon X \to X \times_Y X$ is surjective. The injectivity of f is an easy consequence of this. Since by assumption f is separated, Δ is a closed immersion, thus it is enough to show that all maximal points of $X \times_Y X$ are in its image.

So let $\zeta \in X \times_Y X$ be maximal. Since the projections $X \times_Y X \to X$ and f are flat morphisms, ζ maps to the unique maximal point of X under the projections, and to the unique maximal point in Y under the composition with f. Looking at local rings, we obtain a commutative square

$$\begin{array}{ccc} \mathscr{O}_{X\times_Y X,\zeta} & \longleftarrow & K(X) \\ \uparrow & & \uparrow \\ K(X) & \longleftarrow & K(Y). \end{array}$$

But since the maximal point of X lies in each of the subsets U_i, our assumptions imply that $K(X) = K(Y)$. This implies that the morphism $\operatorname{Spec} \mathscr{O}_{X\times_Y X,\zeta} \to X \times_Y X$ factors through $\operatorname{Spec} K(X) \otimes_{K(Y)} K(X) = \operatorname{Spec} K(X)$ and hence through Δ, which shows that ζ is in the image of Δ. □

(II) With a bit more work, this strategy can be extended to cover all reduced curves, see [GW1] Theorem 15.18.

(III) For the general case, we use the description of S-valued points of projective space in terms of line bundles. We search for a line bundle $\mathscr{M}$

on the given curve C together with a surjection $\mathscr{O}_C^{n+1} \to \mathscr{L}$ such that the corresponding morphism $C \to \mathbb{P}^n_k$ is an immersion.

Let C_{red} be the underlying reduced subscheme of C and by $\iota\colon C_{\mathrm{red}} \to C$ the corresponding closed immersion. By step (II) we know, that on C_{red} a line bundle $\mathscr{L}'$ (together with a surjection $\mathscr{O}_{C_{\mathrm{red}}}^{n'} \to \mathscr{L}'$) with the desired property exists. The crucial step then, is to show that there exists a line bundle $\mathscr{L}$ on C with $\iota^*\mathscr{L} \cong \mathscr{L}'$. One can then show that from such an $\mathscr{L}$ one can construct $\mathscr{M}$ as desired (in fact, this is also an application of cohomological methods, namely "Serre's criterion for ampleness"); we skip this step here.

To proceed, we need the following cohomological description of the Picard group of a scheme.

Proposition 1.4. *Let X be a scheme.*

(1) *Let $\mathscr{U} = (U_i)_i$ be an open cover of X. Then the Čech cohomology group $H^1(\mathscr{U}, \mathscr{O}_X^\times)$ can be identified with the subgroup of* $\mathrm{Pic}(X)$ *consisting of isomorphism classes of line bundles $\mathscr{L}$ such that $\mathscr{L}_{|U_i} \cong \mathscr{O}_{U_i}$ for all i.*

(2) *We have an isomorphism* $\mathrm{Pic}(X) \cong H^1(X, \mathscr{O}_X^\times)$.

The existence of $\mathscr{L}$ follows from the following lemmas together with the Grothendieck vanishing theorem.

Lemma 1.5. *Let X be a scheme. Then there is a natural isomorphism* $\mathrm{Pic}(X) \cong H^1(X, \mathscr{O}_X^\times)$.

Proof. We can compute the H^1 as Čech cohomology. The key point is then the observation that we can identify, for an open cover $\mathscr{U} = (U_i)_{i\in I}$ of X, the Čech cohomology group $H^1(\mathscr{U}, \mathscr{O}_X^\times)$ and the subgroup of the Picard group given by isomorphism classes of line bundles $\mathscr{L}$ on X such that $\mathscr{L}_{|U_i} \cong \mathscr{O}_{U_i}$ for every i. Cf. [GW1] Sections (11.5), (11.7). □

Lemma 1.6. *Let $i\colon X_0 \to X$ be a closed immersion of schemes defined by a quasi-coherent ideal $\mathscr{I} \subset \mathscr{O}_X$ with $\mathscr{I}^2 = 0$ so that we can view $\mathscr{I}$ as $\mathscr{O}_{X_0}$-module. Then there exists an exact sequence of abelian groups*

$$H^1(X_0, \mathscr{I}) \to H^1(X, \mathscr{O}_X^\times) \to H^1(X_0, \mathscr{O}_{X_0}^\times) \to H^2(X_0, \mathscr{I}). \tag{1.0.1}$$

Proof. See Problem sheet 1. □

See [GW2] Theorem 26.16 for references to the missing pieces.

Remark: Projectivity of surfaces 1.7. Let k be a field, and let X be a separated k-scheme of finite type of dimension 2 (i.e., all local rings at closed points have dimension 2). We call X a *surface*.

Theorem. If X is regular (i.e., all local rings of X are regular local rings), then X can be embedded into some projective space over k as a locally closed subscheme.

To prove the theorem, by "Nagata's compactification theorem" (itself a difficult theorem, see [GW1] Section (12.15) for its statement and references) one can restrict to the case that X is a proper k-scheme.

For X proper, a key ingredient is the Lemma of Enriques-Severi-Zariski that we have seen at the end of the Algebraic Geometry 2 class. See [GW2] Theorem 25.151 for a proof of the theorem (which requires quite a few other ingredients, many of them also relying on cohomology).

The regularity assumption cannot be dropped in the theorem. In higher dimension, the corresponding statement fails even for regular k-schemes.

Another example of a topic (among very many others ...) that illustrate the use of "heavy machinery" (in particular, cohomological methods) are the *Weil conjecturs for curves over finite fields*, see [GW2] Sections (26.28), (26.29).

2. Smoothness and differentials

Oct. 16, 2023

General reference: [GW1] Ch. 6.

The Zariski tangent space.

(2.1) Definition of the Zariski tangent space.

Definition 2.1. *Let X be a scheme, $x \in X$, $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ the maximal ideal in the local ring at x, $\kappa(x)$ the residue class field of X in x. The $\kappa(x)$-vector space $(\mathfrak{m}/\mathfrak{m}^2)^*$ is called the* (Zariski) tangent space of X in x.

Definition 2.2. *Let R be a ring, $f_1, \dots, f_r \in R[T_1, \dots, T_n]$. We call the matrix*

$$J_{f_1,\dots,f_r} := \left(\frac{\partial f_i}{\partial T_j}\right)_{i,j} \in M_{r\times n}(R[T_\bullet])$$

the Jacobian matrix *of the polynomials f_i. Here the partial derivatives are to be understood in a formal sense.*

Remark 2.3.
(1) If in the above setting the ideal $\mathfrak{m}$ is finitely generated, then $\dim_{\kappa(x)} T_xX$ is the minimal number of elements needed to generate $\mathfrak{m}$ and in particular is finite.
(2) The tangent space construction is functorial in the following sense: Given a scheme morphism $f\colon X \to Y$ and $x \in X$ such that $\dim_{\kappa(f(x))} T_{f(x)}Y$ is finite or $[\kappa(x) : \kappa(f(x))]$ is finite, then we obtain a map

$$df_x\colon T_xX \to T_{f(x)}Y \otimes_{\kappa(f(x))} \kappa(x).$$

Example 2.4. Let k be a field, $X = V(f_1, \dots, f_m) \subseteq \mathbb{A}^n_k$, $f_i \in k[T_1, \dots, T_n]$, $x = (x_i)_i \in k^n = \mathbb{A}^n(k)$. Then there is a natural identification $T_xX = \operatorname{Ker}(J_{f_1,\dots,f_m}(x))$, where $J_{f_1,\dots,f_m}(x)$ denotes the matrix with entries in $\kappa(x) = k$ obtained by mapping each entry of $J_{f_1,\dots,f_m}$ to $\kappa(x)$, which amounts to evaluating these polynomials at x.

Proposition 2.5. *Let k be a field, X a k-scheme, $x \in X(k)$. There is an identification (functorial in (X, x))*

$$X(k[\varepsilon]/(\varepsilon^2))_x := \{f \in \operatorname{Hom}_k(\operatorname{Spec} k[\varepsilon]/(\varepsilon^2), X);\ \operatorname{im}(f) = \{x\}\} = T_xX.$$

Remark 2.6.
(1) It is possible to define the k-vector space structure on $X(k[\varepsilon]/(\varepsilon^2))_x$ "directly".
(2) Similarly, one can define the relative tangent space of an S-scheme X in a K-valued point ξ for any field K and without restrictions on the residue class field of the image point of ξ, as the set of S-morphisms $f\colon \operatorname{Spec} K[\varepsilon]/(\varepsilon^2) \to X$ with $\operatorname{im}(f) = \operatorname{im}(\xi)$ (and again, this set can be

made into a K-vector space). This concept is sometimes useful, but the result is in general different from the Zariski tangent space.

Smooth morphisms.

(2.2) Dimension of schemes.

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Reference: [GW1] Sections (5.3) ff.

Recall from commutative algebra that for a ring R we define the (Krull) dimension $\dim R$ of R as the supremum over all lengths of chains of prime ideals, or equivalently as the dimension of the topological space $\operatorname{Spec} R$ in the sense of the following definition.

Definition 2.7. *Let X be a topological space. We define the dimension of X as*

$$\dim X := \sup\{\ell;\ \text{there exists a chain } Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_\ell \\ \text{of closed irreducible subsets } Z_i \subseteq X\}.$$

We will use this notion of dimension for non-affine schemes, as well. Recall the following theorem about the dimension of finitely generated algebras over a field from commutative algebra:

Theorem 2.8. *Let k be a field, and let A be a finitely generated k-algebra which is a domain. Let $\mathfrak{m} \subset A$ be a maximal ideal. Then*

$$\dim A = \operatorname{trdeg}_k(\operatorname{Frac}(A)) = \dim A_{\mathfrak{m}}.$$

By passing to an affine cover, we obtain the following corollary:

Corollary 2.9. *Let k be a field, and let X be an integral k-scheme which is of finite type over k. Denote by $K(X)$ its field of rational functions. Let $U \subseteq X$ be a non-empty open subset, and let $x \in X$ be a closed point. Then*

$$\dim X = \dim U = \operatorname{trdeg}_k(K(X)) = \dim \mathscr{O}_{X,x}.$$

(2.3) Definition of smooth morphisms.

Reference: [GW1] Section (6.8).

Definition 2.10. *A morphism $f\colon X \to Y$ of schemes is called* smooth of relative dimension $d \geq 0$ in $x \in X$, *if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \operatorname{Spec} R \subseteq Y$ of $f(x)$ such that $f(U) \subseteq V$ and an open immersion $j\colon U \to \operatorname{Spec} R[T_1, \dots, T_n]/(f_1, \dots, f_{n-d})$ such that the triangle*

$$\begin{array}{ccc} U & \xrightarrow{\ j\ } & \operatorname{Spec} R[T_1, \dots, T_n]/(f_1, \dots, f_{n-d}) \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & V & \end{array}$$

is commutative, and that the Jacobian matrix $J_{f_1,\dots,f_{n-d}}(x)$ *has rank* $n-d$.

We say that $f\colon X \to Y$ *is* smooth of relative dimension d *if* f *is smooth of relative dimension* d *at every point of* X. *Instead of* smooth of relative dimension 0, *we also use the term* étale.

With notation as above, if f is smooth at $x \in X$, then x has an open neighborhood such that f is smooth at all points of this open neighborhood. Clearly, $\mathbb{A}^n_S$ and $\mathbb{P}^n_S$ are smooth of relative dimension n for every scheme S. (It is harder to give examples of non-smooth schemes directly from the definition; we will come back to this later.)

Remark 2.11. (The Jacobian Conjecture) Let k be a field, $n \geq 1$, and let $f_1, \dots, f_n \in k[X_1, \dots, X_n]$. The f_i define a k-scheme morphism $\mathbb{A}^n_k \to \mathbb{A}^n_k$, given on R-valued points by $(x_1, \dots, x_n) \mapsto (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots x_n))$.

Assume that f is an isomorphism of k-schemes. It then follows easily, by similar computations as above (or expressed differently by the "multi-variable chain rule"), that the Jacobian matrix of the f_i is invertible in $\mathrm{Mat}_{n\times n}(k[X_\bullet])$. Equivalently, the determinant of the Jacobian matrix lies in $k^\times$.

Jacobian conjecture (O. Keller, 1939) Let k be a field of characteristic 0, $n \geq 1$, and let $f_1, \dots, f_n \in k[X_1, \dots, X_n]$. The morphism $\mathbb{A}^n_k \to \mathbb{A}^n_k$ induced by the f_i is an isomorphism if and only if the Jacobian matrix $\left(\frac{\partial f_i}{\partial X_i}\right)_{i,j} \in \mathrm{Mat}_{n\times n}(k[X_\bullet])$ is invertible.

For $n = 1$ the statement is easy to prove, but the conjecture is open even for $n = 2$ and is particularly well-known for the number of incorrect attempts of proving it.

It is not very hard to see that the condition that k has characteristic 0 cannot be dropped. Can you find an example for this?

With a bit of effort, one can show that equivalently, one can formulate the conjecture as follows: Let k be a field of characteristic 0, and let $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ be an étale morphism. Then f is an isomorphism.

(2.4) Existence of smooth points.

Reference: [GW1] Section (6.9).

Let k be a field.

Lemma 2.12. ([GW1] Lemma 6.17, Prop. 10.52) *Let* X, Y *be [integral*[1]*] k-schemes which are locally of finite type over k. Let $x \in X$, $y \in Y$, and let $\varphi\colon \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ be an isomorphism of k-algebras. Then there exist open neighborhoods U of x and V of y and an isomorphism $h\colon U \to V$ of k-schemes with $h^\sharp_x = \varphi$.*

[1]The statement is true in general, but in the lecture we proved it only with the additional assumption that X and Y are integral.

Proposition 2.13. *Let X be an integral k-scheme of finite type. Assume that $K(X) \cong k(T_1, \dots, T_d)[\alpha]$ with α separable algebraic over $k(T_1, \dots, T_d)$. (This is always the case, if k is perfect.) (Then $\dim X = d$ by the above.)*

Then there exists a dense open subset $U \subseteq X$ and a separable irreducible polynomial $g \in k(T_1, \dots, T_d)[T]$ with coefficients in $k[T_1, \dots, T_d]$, such that U is isomorphic to a dense open subset of $\operatorname{Spec} k[T_1, \dots T_d]/(g)$.

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Theorem 2.14. *Let k be a perfect field, and let X be a nonempty reduced k-scheme locally of finite type over k. Then the smooth locus*

$$X_{\mathrm{sm}} := \{x \in X;\ X \to \operatorname{Spec} k \text{ is smooth at } x\}$$

of X is open and dense.

(2.5) Regular rings.

Definition 2.15. *A noetherian local ring A with maximal ideal $\mathfrak{m}$ and residue class field κ is called* regular*, if $\dim A = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.*

One can show that the inequality $\dim A \leq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ always holds. Therefore we can rephrase the definition as saying that A is regular if $\mathfrak{m}$ has a generating system consisting of $\dim A$ elements.

Definition 2.16. *A noetherian ring A is called* regular*, if $A_\mathfrak{m}$ is regular for every maximal ideal $\mathfrak{m} \subset A$.*

We quote the following (mostly non-trivial) results about regular rings. A key input for Part (4) is a version of Krull's Principal Ideal Theorem.

Theorem 2.17. (See [GW1] Proposition B.77 for precise references, [M2], [AM] Ch. 11)

(1) *Every localization of a regular ring is regular.*
(2) *If A is regular, then the polynomial ring $A[T]$ is regular.*
(3) *(Theorem of Auslander–Buchsbaum) Every regular local ring is a unique factorization domain.*
(4) *Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and of dimension d, and let $f_1, \dots, f_r \in \mathfrak{m}$. Then $A/(f_1, \dots, f_r)$ is regular of dimension $d - r$ if and only if the images of the f_i in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $A/\mathfrak{m}$.*

Note that Part (3) implies in particular that every regular local ring is a domain. The UFD property also implies that this domain is integrally closed in its field of fractions.

(2.6) Smoothness and regularity.

Reference: [GW1] Section (6.12).

Let k be a field.

Lemma 2.18. *Let X be a k-scheme locally of finite type. Let $x \in X$ such that $X \to \operatorname{Spec} k$ is smooth of relative dimension d in x. Then $\mathscr{O}_{X,x}$ is regular of dimension $\le d$. If moreover x is closed, then $\mathscr{O}_{X,x}$ is regular of dimension d.*

Sketch of proof. First, reduce to the case that (1) x is a closed point in X. By the definition of smooth morphisms, it is then enough to consider the case of a closed point $x \in \operatorname{Spec} k[X_\bullet](f_\bullet)$ where the Jacobian matrix has full rank. By Theorem 2.17 (2) and (4) it is enough to show that the images of the f_i in $\mathfrak{m}_x/\mathfrak{m}_x^2$ are linearly independent. This is clear (cf. Example 2.4) if $\kappa(x) = k$, and the general case can be reduced to this one, using that in the base change $X \otimes_k \kappa(x)$ there exists a point $\overline{x}$ with residue class field $\kappa(x)$ projecting to $x \in X$ and that we have an inclusion $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_{\overline{x}}/\mathfrak{m}_{\overline{x}}^2$ of $\kappa(x)$-vector spaces. □

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Lemma 2.19. *Let $X = V(g_1, \dots, g_s) \subseteq \mathbb{A}_k^n$, and let $x \in X$ be a closed point. If $\operatorname{rk} J_{g_1,\dots,g_s}(x) \ge n - \dim \mathscr{O}_{X,x}$, then x is smooth at X/k, and $\operatorname{rk} J_{g_1,\dots,g_s}(x) = n - \dim \mathscr{O}_{X,x}$.*

Sketch of proof. Write $d = \dim \mathscr{O}_{X,x}$. After renumbering the g_i, if necessary, we may assume that the first $n-d$ columns of $J_{g_\bullet}(x)$ are linearly independent. We then have

$$x \in X = V(g_1, \dots, g_s) \subseteq Y := V(g_1, \dots, g_{n-d}) \subseteq \mathbb{A}_k^n,$$

and x is smooth over k as a point of Y. By the previous lemma, $\dim \mathscr{O}_{Y,x} = d$. It follows that $\mathscr{O}_{X,x} = \mathscr{O}_{Y,x}$, and together with Lemma 2.12 we obtain the claim. □

Lemma 2.20. ([GW1] Corollary 5.47) *Let X be a k-scheme locally of finite type and let $x \in X$ be a closed point. Fix an algebraically closed extension field K of k and write $X_K = X \otimes_k K$. If $\overline{x} \in X_K$ is a point mapping to x, then*

$$\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{X_K,\overline{x}}.$$

Very sketchy indications of proof. For $\ge$ choose some affine open neighborhood of x, apply Noether normalization, and use that the properties *finite* and *injective* of a ring homomorphism are preserved under the base change $- \otimes_k K$.

For $\le$, use that the morphism $X_K \to X$ (being obtained by base change from $\operatorname{Spec}(K) \to \operatorname{Spec}(k)$) is flat, and that flat ring homomorphisms satisfy a *going down* theorem. The key fact for the going down property is that for every flat local ring homomorphism $A \to B$ between local rings, the map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective. Cf. [GW1] Lemma 14.9 or [M2] Theorem 7.3, Theorem 9.5. (In [GW1], the proof of $\le$ is given using the more difficult Proposition 5.44/Theorem 14.38 there, which is required in the book anyway; but at this point the above, related but simpler method works.) □

Theorem 2.21. *Let X be a k-scheme locally of finite type, $x \in X$ a closed point, $d \geq 0$. Fix an algebraically closed extension field K of k and write $X_K = X \otimes_k K$. The following are equivalent:*
(i) *The morphism $X \to \operatorname{Spec} k$ is smooth of relative dimension d at x.*
(ii) *For all points $\overline{x} \in X_K$ lying over x, X_K is smooth over K of relative dimension d at $\overline{x}$.*
(iii) *There exists a point $\overline{x} \in X_K$ lying over x, such that X_K is smooth over K of relative dimension d at $\overline{x}$.*
(iv) *For all points $\overline{x} \in X_K$ lying over x, the local ring $\mathscr{O}_{X_K,\overline{x}}$ is regular of dimension d.*
(v) *There exists a point $\overline{x} \in X_K$ lying over x, such that the local ring $\mathscr{O}_{X_K,\overline{x}}$ is regular of dimension d.*

If these conditions hold, then the local ring $\mathscr{O}_{X,x}$ is regular of dimension d, and if $\kappa(x) = k$, then this last condition is equivalent to the previous ones.

Sketch of proof. The implications (i) ⇒ (ii) ⇒ (iii) and (iv) ⇒ (v) are easy.

Furthermore (iii) ⇒ (iv) and the regularity of $\mathscr{O}_{X,x}$ for a smooth point x follow from Lemma 2.18.

Next we show that the regularity of $\mathscr{O}_{X,x}$ implies that x is a smooth point if $\kappa(x) = k$. Write $d = \dim \mathscr{O}_{X,x} = \dim_k T_x X$. We embed an affine open neighborhood U into affine space, say as an open subscheme of $V(g_1, \dots, g_s) \subseteq \mathbb{A}^n_k$. We are then in the situation of Lemma 2.19, and the lemma shows that x is a smooth point. This also shows (v) ⇒ (iii).

It remains to prove that (iii) ⇒ (i). It is enough to consider the case where x is a closed point of $V(g_1, \dots, g_s) \subset \mathbb{A}^n_k$ for some polynomials g_i. By Lemma 2.19, it is enough to show that $\operatorname{rk} J_{g_\bullet}(x) = n - \dim \mathscr{O}_{X,x}$. But the rank of the Jacobian matrix does not change when we replace x by $\overline{x}$ (and consider the polynomials g_i in $K[X_\bullet]$), and $\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{X_K,\overline{x}}$. Since $\overline{x}$ is a regular point of X_K by (iii), which we now assume to hold, we are done. □

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Corollary 2.22. *Let X be an irreducible scheme of finite type over k, and let $x \in X(k)$ be a k-valued point. Then $X \to \operatorname{Spec} k$ is smooth at x if and only if $\dim X = \dim_k T_x X$.*

Corollary 2.23. *Let $X = V(g_1, \dots, g_s) \subseteq \mathbb{A}^n_k$ and let $x \in X$ be a smooth closed point. Let $d = \dim \mathscr{O}_{X,x}$. Then $J_{g_1,\dots,g_s}(x)$ has rank $n-d$. In particular, $s \geq n - d$.*

After renumbering the g_i, if necessary, there exists an open neighborhood U of x and an open immersion $U \subseteq V(g_1, \dots, g_{n-d})$, i.e., locally around x, "X is cut out in affine space by the expected number of equations".

Corollary 2.24. *Let X be locally of finite type over k. The following are equivalent:*
(i) *X is smooth over k.*
(ii) *$X \otimes_k L$ is regular for every field extension L/k.*

(iii) *There exists an algebraically closed extension field K of k such that $X \otimes_k K$ is regular.*

The sheaf of differentials.

General references: [GW2] Ch. 17, [M2] §25, [Bo] Ch. 8, [H] II.8.

We now introduce the "module of differentials" of a ring homomorphism (and its sheaf version $\Omega_{X/S}$ for a scheme morphism $X \to S$). This allows us to study how the (co-)tangent space varies in a family; as we will see, under suitable assumptions the fiber $\Omega_{X/S}(x)$ at $x \in X$ can be identified with the dual $\mathfrak{m}_x/\mathfrak{m}_x^2$ of T_xX, see Proposition 2.44. The theory we will set up is also closely related to the so-called infinitesimal lifting criterion for smooth morphisms, see Theorem 2.54.

(2.7) Modules of differentials.

Let A be a ring.

Definition 2.25. *Let B be an A-algebra, and M a B-module. An A-*derivation *from B to M is a homomorphism $D\colon B \to M$ of abelian groups such that*
(a) *(Leibniz rule) $D(bb') = bD(b') + b'D(b)$ for all $b, b' \in B$,*
(b) *$d(a) = 0$ for all $a \in A$.*

Assuming property (a), property (b) is equivalent to saying that D is a homomorphism of A-modules. We denote the set of A-derivations $B \to M$ by $\mathrm{Der}_A(B, M)$; it is naturally a B-module.

Definition 2.26. *Let B be an A-algebra. We call a B-module $\Omega_{B/A}$ together with an A-derivation $d_{B/A}\colon B \to \Omega_{B/A}$ a module of (relative, Kähler) differentials of B over A if it satisfies the following universal property:*

For every B-module M and every A-derivation $D\colon B \to M$, there exists a unique B-module homomorphism $\psi\colon \Omega_{B/A} \to M$ such that $D = \psi \circ d_{B/A}$.

In other words, the map $\mathrm{Hom}_B(\Omega_{B/A}, M) \to \mathrm{Der}_A(B, M)$, $\psi \mapsto \psi \circ d_{B/A}$ is a bijection.

Lemma 2.27. *Let I be a set, $B = A[T_i, i \in I]$ the polynomial ring. Then $\Omega_{B/A} := B^{(I)}$ with $d_{B/A}(T_i) = e_i$, the "i-th standard basis vector" is a module of differentials of B/A.*

So we can write $\Omega_{B/A} = \bigoplus_{i\in I} Bd_{B/A}(T_i)$.

Lemma 2.28. *Let $\varphi\colon B \to B'$ be a surjective homomorphism of A-algebras, and write $\mathfrak{b} = \mathrm{Ker}(\varphi)$. Assume that a module of differentials $(\Omega_{B/A}, d_{B/A})$*

for B/A exists. Then

$$\Omega_{B/A}/(\mathfrak{b}\Omega_{B/A} + B'd(\mathfrak{b}))$$

together with the derivation $d_{B'/A}$ induced by $d_{B/A}$ is a module of differentials for B'/A.

Corollary 2.29. *For every A-algebra B, a module $\Omega_{B/A}$ of differentials exists. It is unique up to unique isomorphism.*

We will see later that for a scheme morphism $X \to Y$, one can construct an $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with a "derivation" $\mathscr{O}_X \to \Omega_{X/Y}$ by gluing sheaves associated to modules of differentials attached to the coordinate rings of suitable affine open subschemes of X and Y.

Let $\varphi\colon A \to B$ be a ring homomorphism. For the next definition, we will consider the following situation: Let C be a ring, $I \subseteq C$ an ideal with $I^2 = 0$, and let

$$\begin{array}{ccc} C/I & \longleftarrow & B \\ \uparrow & & \uparrow{\scriptstyle\varphi} \\ C & \longleftarrow & A \end{array}$$

be a commutative diagram (where the right vertical arrow is the canonical projection). We will consider the question whether for these data, there exists a homomorphism $B \to C$ (dashed in the following diagram) making the whole diagram commutative:

$$\begin{array}{ccc} C/I & \longleftarrow & B \\ \uparrow & \swarrow & \uparrow{\scriptstyle\varphi} \\ C & \longleftarrow & A. \end{array}$$

Definition 2.30. *Let $\varphi\colon A \to B$ be a ring homomorphism.*

(1) *We say that φ is* formally unramified*, if in every situation as above, there exists at most one homomorphism $B \to C$ making the diagram commutative.*
(2) *We say that φ is* formally smooth*, if in every situation as above, there exists at least one homomorphism $B \to C$ making the diagram commutative.*
(3) *We say that φ is* formally étale*, if in every situation as above, there exists a unique homomorphism $B \to C$ making the diagram commutative.*

Passing to the spectra of these rings, we can interpret the situation in geometric terms: $\operatorname{Spec} C/I$ is a closed subscheme of $\operatorname{Spec} C$ with the same topological space, so we can view the latter as an "infinitesimal thickening" of the former. The question becomes the question whether we can extend the morphism from $\operatorname{Spec} C/I$ to $\operatorname{Spec} B$ to a morphism from this thickening.

Construction 2.31. Let B be a ring and let M be a B-module. We construct a B-algebra $D_B(M)$ as follows. As additive groups, we set $D_B(M) = B \times M$. The multiplication is defined by

$$(b, m)(b', m') = (bb', bm' + b'm).$$

Then $M = \{0\} \times M \subseteq D_B(M)$ is an ideal with $M^2 = 0$.

For example, taking $M = B$, we have $D_B(B) \cong B[\varepsilon](\varepsilon^2)$, the ring of dual numbers over B.

The projection $\pi\colon D_B(M) \to B$ is a B-algebra homomorphism, i.e., the composition $B \to D_B(M) \to B$ is the identity.

Now suppose that B is an A-algebra. One then checks that the map

$$\mathrm{Der}_A(B, M) \to \{\psi \in \mathrm{Hom}_A(B, D_B(M));\ \pi\circ\psi = \mathrm{id}_B\}, \quad D \mapsto (b \mapsto (b, D(b)),$$

is a B-module isomorphism.

Proposition 2.32. *Let $\varphi\colon A \to B$ be a ring homomorphism. Then φ is formally unramified if and only if $\Omega_{B/A} = 0$.*

Proof. Assume that $\Omega_{B/A} = 0$, and consider $I \subset C$ and a commutative diagram as above. We need to show that there is at most one ring homomorphism $B \to C$ making the diagram commutative. Assume that $\varphi_1, \varphi_2\colon B \to C$ have this property. The C-module structure on I factors through a C/I-module structure since $I^2 = 0$, so that we can view I as a B-module via the map $B \to C/I$. Then the difference $\varphi_1 - \varphi_2$ is an A-derivation $B \to I$, and is hence zero by our assumption.

For the converse it is enough that every A-derivation $B \to M$ vanishes. Let $C = D_B(M)$ and $I = M$. Then $I^2 = 0$, and the assumption that B is formally unramified over A implies $\mathrm{Der}_A(B, M) = 0$. □

For an algebraic field extension L/K one can show that $K \to L$ is formally unramified if and only if it is formally smooth if and only if L/K is separable. Cf. Problem 27 and [M2] §25, §26 (where the discussion is extended to the general, not necessarily algebraic, case). Nov. 6, 2023

Theorem 2.33. *Let $f\colon A \to B$, $g\colon B \to C$ be ring homomorphisms.*

(1) *Then we obtain a natural sequence of C-modules*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

which is exact.

(2) *If moreover g is formally smooth, then the sequence*

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

is a split short exact sequence.

(3) *Conversely, assume that $g \circ f$ is formally smooth and that the sequence*

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

is a split short exact sequence. Then g is formally smooth.

Proof. To check the exactness in Part (1), it is enough to check that the sequence gives rise to an exact sequence whenever we apply the functor $\mathrm{Hom}_C(-, M)$ for M a C-module. Note that $\mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, M) = \mathrm{Hom}_B(\Omega_{B/A}, M)$ (where on the right we view M as a B-module via g). See [ALG2] Satz 3.14.

Thus the first part follows, once we check that

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M)$$

is exact (as a sequence of A-modules or just abelian groups) for any C-module M. But this is obvious.

Part (2). Now assume that g is formally smooth. Let us construct a C-module homomorphism $\Omega_{C/A} \to \Omega_{B/A} \otimes_B C$ as follows. Constructing a homomorphism like this amounts to constructing an A-derivation $C \to \Omega_{B/A} \otimes_B C =: M$. Similarly as above, we consider $C \times M$ as a ring (with $M^2 = 0$). Let $B \to C \times M$ be given by $b \mapsto (g(b), db \otimes 1)$. One checks that this is a ring homomorphism. Since g is formally smooth, for this B-algebra structure we find a homomorphism $C \to C \times M$ of B-algebras. Composing it with the projection to M we obtain an A-derivation $C \to M$. One checks that the composition $\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{B/A} \otimes_B C$ is the identity, and this finishes the proof.

See also [GW2] Proposition 18.18 (1) for a slightly different proof of the second part (which is more along the lines of our proof of the first part).

Part (3). This can be proved by similar arguments as for Parts (1) and (2). We omit the proof for the time being (see [GW2] Proposition 18.18 (2)). □

Theorem 2.34. *Let $f\colon A \to B$, $g\colon B \to C$ be ring homomorphisms. Assume that g is surjective with kernel $\mathfrak{b}$.*

(1) *There is a natural sequence of C-modules*

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0,$$

where the homomorphism $\mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B C$ is given by $x \mapsto d_{B/A}(x) \otimes 1$.

(2) *If moreover $g \circ f$ is formally smooth, then the sequence*

$$0 \to \mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

is a split short exact sequence.

Proof. All assertions in Part (1) follow from Theorem 2.33 and Lemma 2.28.

To prove Part (2), consider the short exact sequence

$$0 \to \mathfrak{b}/\mathfrak{b}^2 \to B/\mathfrak{b}^2 \xrightarrow{p} C \to 0.$$

The assumption that $g \circ f$ is formally smooth implies that p admits a section s. Then $s \circ p_{|\mathfrak{b}/\mathfrak{b}^2} = 0$, and $p \circ (\mathrm{id} - s \circ p) = 0$. We obtain $D := \mathrm{id} - s \circ p\colon B/\mathfrak{b}^2 \to \mathfrak{b}/\mathfrak{b}^2$. This is an element of $\mathrm{Der}_A(B/\mathfrak{b}^2, \mathfrak{b}/\mathfrak{b}^2) = \mathrm{Hom}_B(\Omega_{B/A}, \mathfrak{b}/\mathfrak{b}^2)$ and one checks that it gives rise to a retraction of the map $\mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B C$ in the sequence in Part (2). □

(2.8) The sheaf of differentials of a scheme morphism.

Remark 2.35. Let again B an A-algebra. There is the following alternative construction of $\Omega_{B/A}$: Let $m\colon B \otimes_A B \to B$ be the multiplication map, and let $I = \mathrm{Ker}(m)$. Then I/I^2 is a B-module, and $d\colon B \to I/I^2$, $b \mapsto 1\otimes b - b\otimes 1$, is an A-derivation.

Let us show that $(I/I^2, d)$ satisfies the universal property defining $(\Omega_{B/A}, d_{B/A})$. Let M be a B-module. Composition with d gives a map $\mathrm{Hom}_A(I/I^2, M) \to \mathrm{Der}_A(B, M)$. To show that it is injective, it is enough to show that I/I^2 is generated by the image of d as a B-module. This follows from the following two computations (for $b, b', b_i, b'_i \in B$):
(1) $b \otimes b' = bb' \otimes 1 + (b \otimes 1)(1 \otimes b' - b' \otimes 1)$,
(2) if $\sum b_i b'_i = 0$, then $\sum b_i \otimes b'_i = \sum (b_i \otimes 1)(1 \otimes b'_i - b'_i \otimes 1)$ by (1).

For the surjectivity, let $D \in \mathrm{Der}_A(B, M)$ and let $\psi\colon B \to D_B(M)$, $b \mapsto (b, D(b))$, the corresponding map, cf. Construction 2.31. The diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I/I^2 & \longrightarrow & B \otimes B/I^2 & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & \big\downarrow {\scriptstyle b\otimes b' \mapsto b\psi(b')} & & \big\downarrow {\scriptstyle =} & & \\
0 & \longrightarrow & M & \longrightarrow & D_B(M) & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

(with exact rows) induces a map $\delta\colon I/I^2 \to M$ which makes the whole diagram commute, and $\delta \circ d = D$.

Nov. 8, 2023

There are several ways of defining an $\mathscr{O}_X$-module of differentials for a morphism $f\colon X \to Y$ of schemes. One way is to proceed by gluing, using the following remark.

Remark 2.36. Let $A \to B$ be a ring homomorphism, and let $S \subseteq B$ be a multiplicative subset. Then there is a natural identification $S^{-1}\Omega_{B/A} = \Omega_{N/A} \otimes_B S^{-1}B = \Omega_{S^{-1}/A}$. If $T \subseteq A$ is a multiplicative subset that is mapped to $(S^{-1}B)^\times$ under the natural homomorphism $A \to B \to S^{-1}B$, then this module can also be identified with $\Omega_{S^{-1}B/T^{-1}A}$.

To pin down the sheaf of differentials we first define the notion of derivation in this context.

Definition 2.37. *Let $X \to Y$ be a morphism of schemes, and let $\mathscr{M}$ be an $\mathscr{O}_X$-module. A derivation $D\colon \mathscr{O}_X \to \mathscr{M}$ is a homomorphism of abelian sheaves such that for all open subsets $U \subseteq X$, $V \subseteq Y$ with $f(U) \subseteq V$, the map $\mathscr{O}(U) \to \mathscr{M}(U)$ is an $\mathscr{O}_Y(V)$-derivation.*

Equivalently, $D\colon \mathscr{O}_X \to \mathscr{M}$ is a homomorphism of $f^{-1}(\mathscr{O}_Y)$-modules such that for every open $U \subseteq X$, the Leibniz rule

$$D(U)(bb') = bD(U)(b') + b'D(U)(b), \qquad \forall b, b' \in \Gamma(U, \mathscr{O}_X)$$

holds.

We denote the set of all these derivations by $\mathrm{Der}_Y(\mathscr{O}_X, \mathscr{M})$; it is a $\Gamma(X, \mathscr{O}_X)$-module.

Definition/Proposition 2.38. *Let $f\colon X \to Y$ be a morphism of schemes. The following three definitions give the same result (up to unique isomorphism), called the sheaf of differentials of f or of X over Y, denoted $\Omega_{X/Y}$ — a quasi-coherent $\mathscr{O}_X$-module together with a derivation $d_{X/Y}\colon \mathscr{O}_X \to \Omega_{X/Y}$.*

(i) *There exists a unique $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with a derivation $d_{X/Y}\colon \mathscr{O}_X \to \Omega_{X/Y}$ such that for all affine open subsets $\operatorname{Spec} B = U \subseteq X$, $\operatorname{Spec} A = V \subseteq Y$ with $f(U) \subseteq V$, $\Omega_{X/Y|U} = \widetilde{\Omega_{B/A}}$ and $d_{X/Y|U}$ is induced by $d_{B/A}$.*

(ii) *Define $\Omega_{X/Y} = \Delta^*(\mathscr{J}/\mathscr{J}^2)$, where $\Delta\colon X \to X \times_Y X$ is the diagonal morphism, $W \subseteq X \times_Y X$ is open such that $\operatorname{im}(\Delta) \subseteq W$ is closed (if f is separated we can take $W = X \times_Y X$), and $\mathscr{J}$ is the quasi-coherent ideal defining the closed subscheme $\Delta(X) \subseteq W$. Define the derivation $d_{X/Y}$ as the one induced, on affine opens, by the map $b \mapsto 1 \otimes b - b \otimes 1$.*

(iii) *The quasi-coherent $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with $d_{X/Y}$ is characterized by the universal property that composition with $d_{X/Y}$ induces bijections*

$$\operatorname{Hom}_{\mathscr{O}_X}(\Omega_{X/Y}, \mathscr{M}) \xrightarrow{\sim} \operatorname{Der}_Y(\mathscr{O}_X, \mathscr{M})$$

for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$, functorially in $\mathscr{M}$.

The properties we proved for modules of differentials can be translated into statements for sheaves of differentials. In all statements here, equality means that there is a unique isomorphism that is compatible with the universal derivations.

Proposition 2.39. *Let $f\colon X \to Y$ be a morphism of schemes.*

(1) *Let $g\colon Y' \to Y$ be a morphism of schemes, and let $X' = X \times_Y Y'$. Denote by $g'\colon X' \to X$ the base change of g. There is a natural isomorphism $\Omega_{X'/Y'} = (g')^*\Omega_{X/Y}$.*

(2) *Let $U \subseteq X$ and $V \subseteq V$ be open subsets with $f(U) \subseteq V$. There is a natural identification $\Omega_{X/Y|U} = \Omega_{U/V}$.*

(3) *Let $x \in X$. Then $\Omega_{X/Y,x} = \Omega_{\mathscr{O}_{x,x}/\mathscr{O}_{Y,y}}$.*

We can use a similar definition as we used for ring homomorphisms above to define the notions of formally unramified, formally smooth and formally étale morphisms of schemes.

Definition 2.40. *Let $f\colon X \to Y$ be a morphism of schemes.*

(1) *We say that f is* formally unramified*, if for every ring C, every ideal I with $I^2 = 0$, and every morphism $\operatorname{Spec} C \to Y$ (which we use to view $\operatorname{Spec} C$ and $\operatorname{Spec} C/I$ as Y-schemes), the composition with the natural closed embedding $\operatorname{Spec} C/I \to \operatorname{Spec} C$ yields an injective map $\operatorname{Hom}_Y(\operatorname{Spec} C, X) \to \operatorname{Hom}_Y(\operatorname{Spec} C/I, X)$.*

(2) *We say that f is* formally smooth*, if for every ring C, every ideal I with $I^2 = 0$, and every morphism $\operatorname{Spec} C \to Y$, the composition with the natural closed embedding $\operatorname{Spec} C/I \to \operatorname{Spec} C$ yields a surjective map $\operatorname{Hom}_Y(\operatorname{Spec} C, X) \to \operatorname{Hom}_Y(\operatorname{Spec} C/I, X)$.*

(3) *We say that f is* formally étale, *if f is formally unramified and formally smooth.*

If f is a morphism of affine schemes, then f has one of the properties of this definition if and only if the corresponding ring homomorphism has the same property in the sense of our previous definition.

Proposition 2.41. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms of schemes. Then there is an exact sequence*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

of $\mathscr{O}_X$-modules. If f is formally smooth, then the sequence

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

is exact and splits locally on X.

Proposition 2.42. *Let $i\colon Z \to X$ be a closed immersion with corresponding ideal sheaf $\mathscr{J} \subseteq \mathscr{O}_X$, and let $g\colon X \to Y$ be a scheme morphism. Then there is an exact sequence*

$$i^*(\mathscr{J}/\mathscr{J}^2) \to i^*\Omega_{X/Y} \to \Omega_{Z/Y} \to 0$$

of $\mathscr{O}_Z$-modules. If Z is formally smooth over Y, then the sequence

$$0 \to i^*(\mathscr{J}/\mathscr{J}^2) \to i^*\Omega_{X/Y} \to \Omega_{Z/Y} \to 0$$

is exact and splits locally on Z.

Remark 2.43. When we say that a short exact sequence of $\mathscr{O}_X$-modules *splits locally on a scheme X*, this means that there exists an open cover $X = \bigcup_i U_i$ such that for each i the sequence splits after restricting it to U_i. (It follows from the short exact sequence attached to the local-to-global spectral sequence for Ext sheaves and the vanishing of higher cohomology of quasi-coherent sheaves on affines that a short exact sequence of quasi-coherent $\mathscr{O}_X$-modules that splits locally on X and where the term on the right hand side is of finite presentation, splits on every affine open of X.)

Applying Proposition 2.42 to X a scheme of finite type over $Y = \operatorname{Spec}(k)$, k a field, and $Z = \operatorname{Spec}(k)$ so that i is a k-valued point, we obtain the following description of the fiber of the sheaf of differentials at x.

Proposition 2.44. *Let K be a field, and let X be a k-scheme of finite type. Let $x \in X(k)$. Then we have an isomorphism $T_xX = \Omega_{X/k}(x)^\vee$ between the Zariski tangent space at x and the dual space of the fiber of the sheaf of differentials of X/k at x.*

Similarly, we have the following description. For any scheme Y, we write $Y[\varepsilon] := Y \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]/(\varepsilon^2)$. Denote by $\iota_Y\colon Y \to Y[\varepsilon]$ the natural map. For any morphism $X \to S$ of schemes we write Nov. 13, 2023

$$\mathscr{T}_{X/S} := \mathscr{H}om_{\mathscr{O}_X}(\Omega_{X/S}, \mathscr{O}_X)$$

and call this the *tangent sheaf* of X over S.

Proposition 2.45. *Let $f\colon X \to S$ be a morphism of schemes. For every X-scheme $g\colon Y \to X$, we have a bijection*

$$\mathrm{Hom}^g(Y[\varepsilon], X) := \{\tilde{g}\colon Y[\varepsilon] \to X;\ \tilde{g} \circ \iota_Y = g\} \xrightarrow{\cong} \Gamma(Y, g^*\mathscr{T}_{X/Y}),$$

and these bijections are functorial in Y.

Proof. First note that $\mathrm{Hom}^g(Y[\varepsilon], X)$ can be identified with $\mathrm{Der}_S(\mathscr{O}_X, g_*\mathscr{O}_Y)$. In fact, this can be checked on an affine open cover, and in the affine case we have seen this in Construction 2.31. Now we conclude by the following chain of isomorphisms:

$$\mathrm{Der}_S(\mathscr{O}_X, g_*\mathscr{O}_Y) = \mathrm{Hom}_{\mathscr{O}_X}(\Omega_{X/S}, g_*\mathscr{O}_Y) = \mathrm{Hom}_{\mathscr{O}_Y}(g^*\Omega_{X/S}, \mathscr{O}_Y) = \Gamma(Y, g^*\mathscr{T}_{X/S}).$$

□

Using this description, we "compute" the sheaf of differentials of projective space.

Proposition 2.46. *Let R be a ring. We have a short exact sequence*

$$0 \to \Omega_{\mathbb{P}^n_R/R} \to \mathscr{O}(-1)^{n+1} \to \mathscr{O} \to 0$$

of $\mathscr{O}_X$-modules, called the Euler sequence*.*

Proof. Write $X = \mathbb{P}^n_R$. We have the "universal" surjection $\mathscr{O}_X^{n+1} \to \mathscr{O}_X(1)$ and denote by $\mathscr{K}$ its kernel. We want to show that $\mathscr{K}(-1) := \mathscr{K} \otimes_{\mathscr{O}_X} \mathscr{O}_X(-1) \cong \Omega_{X/R}$. All the $\mathscr{O}_X$-modules involved here are locally free of finite rank, so it is enough to prove that $\mathscr{T}_{X/R} \cong \mathscr{H}om(\mathscr{K}, \mathscr{O}(1)) = \mathscr{H}om(\mathscr{K}(-1), \mathscr{O}) = \mathscr{K}(-1)^\vee$.

Let $U = \mathrm{Spec}(A) \subseteq X$ be open affine and denote by $g\colon U \to X$ the inclusion. The morphism g corresponds to a surjection $A^{n+1} \to L$ onto a locally free A-module L of rank 1 whose kernel we denote by K. Note that $K = \mathscr{K}_{|U}$. We now use the notation of Proposition 2.45.

Claim. There is a natural identification $\mathrm{Hom}_A(K, A^{n+1}/K) \xrightarrow{\cong} \mathrm{Hom}^g(U[\varepsilon], X)$.

Proof of claim. An element of $\mathrm{Hom}^g(U[\varepsilon], X) \subseteq X(U(\varepsilon))$ is given by a surjection $(A[\varepsilon]/(\varepsilon^2))^{n+1} \to L'$, where L' is locally free over $A[\varepsilon]/(\varepsilon^2)$ of rank 1, or equivalently its kernel $K' \subset (A[\varepsilon]/(\varepsilon^2))^{n+1}$, such that $K' \otimes_{A[\varepsilon]/(\varepsilon^2)} A = K$.

Now take an A-module homomorphism $\alpha\colon K \to A^{n+1}/K$. We define K' as the $A[\varepsilon]/(\varepsilon^2)$-module generated by the image of the map $K \to (A[\varepsilon]/(\varepsilon^2))^{n+1}$, $x \to x + \varepsilon\alpha(x)$. (We define $\varepsilon\alpha(x)$ by choosing a lift of $\alpha(x) \in A^{n+1}/K$ in A^{n+1}. The resulting K' is independent of the choice of lift.)

Note that $A[\varepsilon]^{n+1}/K'$ is locally free over $A[\varepsilon]$. To check this, we may localize and thus assume that K and A^{n+1}/K are free A-modules. Now choosing lifts of bases of K and A^{n+1}/K to $A[\varepsilon]^{n+1}$ gives us a family of $n+1$ vectors. Write them as the columns of a matrix M over $A[\varepsilon]$. By construction, $\det(M)$ maps to a unit in A and hence is a unit in $A[\varepsilon]$. Thus the lifts form a basis of $A[\varepsilon]^{n+1}$ and in particular $A[\varepsilon]^{n+1}/K'$ (and K') are free.

This defines the desired bijection.

With the claim and Proposition 2.45 we can identify $\Gamma(U, \mathscr{T}_{X/R})$ with

$$\mathrm{Hom}_A(K, A^{n+1}/K) = \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{K}_{|U}, \mathscr{O}_X(1)_{|U}) = \Gamma(U, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{K}, \mathscr{O}_X(1))).$$

This identification is compatible with restrictions to smaller subsets and therefore defines the isomorphism of $\mathscr{O}_X$-modules we are looking for. □

Remark 2.47.

(1) In the course of the proof we have established a canonical identification of the tangent space $T_x\mathbb{P}^n_k$ of projective space over a field k in a k-valued point x with the vector space $\mathrm{Hom}_k(K, k^{n+1}/K)$, where $K = \mathrm{Ker}(k^{n+1}, L)$ is the kernel of the quotient of k^{n+1} corresponding to x via the functorial description of $\mathbb{P}^n_k$. At this point we use the point of view that $\mathbb{P}^n_k(k)$ is the set of all 1-dimensional quotients of k^{n+1}, or equivalently – passing to the kernel of the projection – of all hyperplanes in k^{n+1}.

Passing to the dual (and classical) point of view, K gives us a line $K^\perp = (k^{n+1}/K)^\vee$ in the dual vector space $(k^{n+1})^\vee$ (which we could identify with k^{n+1} via the standard basis). Then the tangent space is identified with $\mathrm{Hom}_k(K^\perp, k^{n+1,\vee}/K^\perp)$, which is isomorphic to $k^{n+1,\vee}/K^\perp$ since $K^\perp \cong k$. This is "the same" description as using the natural surjection $\mathbb{A}^{n+1}_k \setminus \{0\} \to \mathbb{P}^n_k$ which induces surjections on tangent spaces, cf. [GW1] Prop. 6.10.

(2) As for every short exact sequence of locally free modules of finite rank, we obtain an identification for the top exterior powers,

$$\bigwedge^n \Omega_{\mathbb{P}^n_R/R} \cong \bigwedge^n \Omega_{\mathbb{P}^n_R/R} \otimes \bigwedge^1 \mathscr{O}_{\mathbb{P}^n_R/R} \cong \bigwedge^{n+1} \mathscr{O}(-1)^{n+1} \cong \mathscr{O}(-n-1).$$

Example 2.48. For $n = 1$, the previous proposition gives $\Omega_{\mathbb{P}^1_R/R} \cong \mathscr{O}_{\mathbb{P}^1_R}(-2)$. This is also easy to check directly. The key computation is

$$0 = d(1) = d(\frac{X_i}{X_j}\frac{X_j}{X_i}) = \frac{X_i}{X_j}d\frac{X_j}{X_i} + \frac{X_j}{X_i}d\frac{X_i}{X_j},$$

which implies

$$d\frac{X_j}{X_i} = -\left(\frac{X_j}{X_i}\right)^2 d\frac{X_i}{X_j}.$$

The latter equality describes how $\Omega_{\mathbb{P}^1_R/R}$ is glued from the free modules $\Omega_{\mathbb{P}^1_R/R|D_+(X_i)}$. It coincides with the way we glue to obtain $\mathscr{O}(-2)$.

(2.9) Sheaves of differentials and smoothness.

We start by slightly rephrasing the definition of a smooth morphism. Nov. 15, 2023

Definition 2.49. *A morphism $f\colon X \to Y$ of schemes is called* smooth of relative dimension $d \geq 0$ in $x \in X$, *if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \operatorname{Spec} R \subseteq Y$ of $f(x)$ such that $f(U) \subseteq V$ and an open immersion $j\colon U \to \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d})$ such that the triangle*

$$\begin{array}{ccc} U & \xrightarrow{\quad j \quad} & \operatorname{Spec} R[T_1, \dots, T_n]/(f_1, \dots, f_{n-d}) \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & V & \end{array}$$

is commutative, and that the images of $df_1, \dots, df_{n-d}$ in the fiber $\Omega^1_{\mathbb{A}^n_R/R} \otimes \kappa(x)$ are linearly independent over $\kappa(x)$. (We view x as a point of $\mathbb{A}^n_R$ via the embedding $U \to \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d}) \to \operatorname{Spec} R[T_1, \dots, T_n] = \mathbb{A}^n_R$.)

To see the equivalence, use that $df = \sum_i \frac{\partial f}{\partial X_i} dX_i$.

Proposition 2.50. *Let $f\colon X \to Y$ be smooth of relative dimension d at $x \in X$. Then there exists an open neighborhood U of x such that the restriction $\Omega_{X/Y|U} (= \Omega_{U/Y})$ is free of rank d.*

Proof. Since the assertion is local on X, we may assume that $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} R[T_1, \dots, T_n]/(f_1, \dots, f_{n-d})$ with the $df_i(x) \in \Omega_{\mathbb{A}^n_R/R}(x)$ linearly independent, as in Definition 2.49. We write $\mathfrak{a} = (f_1, \dots, f_{n-d})$ and $A = R[T_1, \dots, T_n]/\mathfrak{a}$. We have the exact sequence (Theorem 2.34)

$$\mathfrak{a}/\mathfrak{a}^2 \to \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \to \Omega_{A/R} \to 0.$$

Renumber the X_i (if necessary) so that the images of $dX_1, \dots, dX_d, df-1, \dots, df_{n-d}$ are a basis of the fiber $(\Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A)(x)$ over x. By the lemma of Nakayama, these elements give us also a basis of the stalk, and hence even a basis on an open neighborhood U of x. The image of $\mathfrak{a}/\mathfrak{a}^2$ is exactly the submodule generated by the df_i, so this implies that $\Omega_{A/R}$ is free over such a neighborhood. □

Remark 2.51. Let us check that in the situation of the previous proposition (and with the notation of its proof), the sequence

$$0 \to \mathfrak{a}/\mathfrak{a}^2 \to \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \to \Omega_{A/R} \to 0.$$

is split exact over U. Since $(\Omega_{A/R})_{|U}$ is free, it is clear that the sequence splits, once we have shown the exactness. Thus it is enough to show that the map on the left hand side is injective. Take $g = \sum c_j f_j \in \mathfrak{a}$, $c_i \in R[T_\bullet]$. Then

$$dg = \sum_j (c_j df_j + f_j dc_j) = \sum_j c_j df_j \in \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A = \Omega_{R[T_\bullet]/R}/\mathfrak{a},$$

so if $dg = 0$, then (after restricting to U, where the df_j are part of a basis) all c_j lie in $\mathfrak{a}$ and hence $g \in \mathfrak{a}^2$.

Theorem 2.52. *Let k be an algebraically closed field, and let X be an irreducible k-scheme of finite type. Let $d = \dim X$. Then X is smooth over k if and only if $\Omega_{X/k}$ is locally free of rank d.*

Proof. If $\Omega_{X/Y}$ is locally free of rank $\dim X$, then X is regular and hence, since k is algebraically closed, also smooth over k (Theorem 2.21). Conversely, the smoothness of f implies that $\Omega_{X/Y}$ is locally free by Proposition 2.50. Again using Theorem 2.21, we also obtain that X is regular, and it follows that f must be smooth of relative dimension $\dim X$. □

Proposition 2.53. *Let $f\colon X \to Y$ be smooth of relative dimension d at $x \in X$. Then there exists an open neighborhood U of x such that the restriction $U \to Y$ of f to U is formally smooth.*

Proof. As in the proof of Proposition 2.50, it is enough to consider the local situation, and we again use the notation set up in the beginning of the proof of that proposition.

Consider a ring C, an ideal I of C with $I^2 = 0$ and a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & C/I \\ \uparrow & & \uparrow \\ R & \longrightarrow & C. \end{array}$$

We need to show that there exists a homomorphism $\varphi\colon A \to C$ making the diagram commutative. We start by choosing arbitarily an R-algebra homomorphism $\psi\colon R[T_1, \dots, T_n] \to C$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & C/I \\ \uparrow & & \uparrow \\ R[T_\bullet] & \longrightarrow & C \end{array}$$

is commutative. Then $\psi(\mathfrak{a}) \subseteq I$ (but of course there is no reason to expect that ψ will factor through A; we will now change it appropriately to achieve that). In Remark 2.51 we have seen that the sequence

$$0 \to \mathfrak{a}/\mathfrak{a}^2 \to \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \to \Omega_{A/R} \to 0.$$

is split exact, at least after replacing A by a suitable localization. Since the proposition makes only a local statement the localization is harmless and we suppress it from the notation. The restriction of ψ to $\mathfrak{a}$ induces a map $\mathfrak{a}/\mathfrak{a}^2 \to I/I^2 = I$, as we have already noted, and since the sequence is split, we can extend that map to a map $\xi\colon \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \to I$. We define D as the composition

$$R[T_\bullet] \xrightarrow{d} \Omega_{R[T_\bullet]/R} \to \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \xrightarrow{\xi} I,$$

an R-derivation with the property that $\psi_{|\mathfrak{a}} = D_{|\mathfrak{a}}$. Setting $\varphi = \psi - D$, we obtain a map that maps $\mathfrak{a}$ to 0 and (since D is a derivation) is a ring homomorphism. Thus φ factors through a homomorphism $\varphi\colon A \to C$. This makes the above diagram commutative, so we are done. □

Theorem 2.54. *Let $f\colon X \to Y$ be a morphism locally of finite presentation (e.g., if Y is noetherian and f is locally of finite type). Then f is smooth if and only if f is formally smooth.*

Proof. Let f be formally smooth and locally of finite presentation. To show that f is smooth, we may work locally on X and Y and therefore pass to an affine situation, i.e., assume that f is given by a ring homomorphism $R \to R[T_\bullet] \to A$ with $R[T_\bullet] \to A$ surjective with kernel $\mathfrak{a}$. Then Theorem 2.34 shows that the sequence

$$0 \to \mathfrak{a}/\mathfrak{a}^2 \to \Omega_{R[T_\bullet]/R} \otimes_{R[T_\bullet]} A \to \Omega_{A/R} \to 0.$$

is split exact. Choosing a basis of $\mathfrak{a}/\mathfrak{a}^2$ and lifting its elements to polynomials $f_1, \dots, f_{n-d} \in \mathfrak{a}$, we see that the conditions of Definition 2.49 are satisfied and the morphism $\operatorname{Spec} A \to \operatorname{Spec} R$ is smooth.

For the converse, note that the previous proposition shows already that a smooth morphism is at least "locally formally smooth". We only give some very sketchy indications on how to get a global version. See [Bo] Ch. 8.5 for more details. See also [GW2] Section (18.10) for a slightly different approach.

Consider a diagram

$$\begin{array}{ccc} \operatorname{Spec} C/I & \xrightarrow{a_0} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} C & \longrightarrow & Y \end{array}$$

where, as usual, $I \subseteq C$ is an ideal with $I^2 = 0$. We have seen that there exists an open cover $X = \bigcup_i U_i$ such that each U_i is formally smooth over Y. In particular after restricting a_0 to U_i and the inverse image of U_i in $\operatorname{Spec} C/I$, we can find the desired diagonal morphism that extends a_0 to $\operatorname{Spec} C$. In other words, we find an open cover $(V_i)_i$ of $\operatorname{Spec} C$ (which topologically is $= \operatorname{Spec} C/I$) and morphisms $\varphi_i\colon V_i \to U_i \subseteq X$ making the above diagram commutative. The idea is to replace the φ_i by φ'_i such that φ'_i and φ'_j coincide on $V_i \cap V_j$. By gluing one obtains the desired map $\operatorname{Spec} C \to X$.

Here, we want to set $\varphi'_i = \varphi_i - D_i$ for some derivation D_i (cf. Construction 2.31 where we have seen this principle). Writing this out one sees that there exists a family $(D_i)_i$ with the desired properties if and only if a certain class (depending on the φ_i) in $\check{H}^1(\operatorname{Spec} C/I, \mathscr{H}om_{\mathscr{O}_{\operatorname{Spec} C/I}}(a_0^*\Omega_{X/Y}, \tilde{I})$ vanishes. But this cohomology group vanishes entirely since $\operatorname{Spec} C/I$ is affine and $\mathscr{H}om_{\mathscr{O}_{\operatorname{Spec} C/I}}(a_0^*\Omega_{X/Y}, \tilde{I})$, $\Omega_{X/Y}$ being of finite presentation by our assumptions, is quasi-coherent. □

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Remark 2.55. We mention the following further facts without proof. See for instance [GW2] Chapter 18.

(1) A morphism of schemes is smooth if and only if it is locally of finite presentation, flat and has regular geometric fibers. (Here the geometric fibers of a morphism $X \to Y$ are the schemes $X \times_Y \operatorname{Spec}(K)$ where K is an algebraically closed field and the fiber product is taken with respect to f and a K-valued point of Y.) This also gives us a "fiber criterion for smoothness", cf. [GW2] Corollary 18.77.
(2) A morphism of schemes is étale (which we have defined as smooth of relative dimension 0) if and only if it is locally of finite presentation and formally étale if and only if it is flat and unramified.
(3) An étale morphism is locally standard-étale ([GW2] Theorem 18.42): For $f\colon X \to Y$ locally of finite presentation and $x \in X$, $y = f(x)$, f is étale at x if and only if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \operatorname{Spec} R \subseteq Y$ of y where $U \cong \operatorname{Spec}(R[T]/(f))_g$ with $f, g \in R[T]$ and f' a unit in the localization $R[T]_g$.
(4) If $f\colon X \to Y$ is smooth at $x \in X$, then there exists an open neighborhood U of x such that $f_{|U}$ can be factorized as $U \to \mathbb{A}^n_Y \to Y$ with $U \to \mathbb{A}^n_Y$ étale.

3. Serre duality

General references: [GW2] Ch. 25 and the references given there; [AK], [H] III.7.

We now come back to Serre duality. The explicit computation of the cohomology groups $H^i(\mathbb{P}^n_R, \mathscr{O}(d))$, R some ring, that we have done in *Algebraic Geometry 2* shows that for every line bundle $\mathscr{L}$ on $X := \mathbb{P}^n_R$ we have a perfect pairing

$$H^i(X, \mathscr{L}) \times H^{n-i}(X, \mathscr{L}^\vee \otimes \omega_X) \to H^n(X, \omega_X) = R.$$

Here $\omega_X := \mathscr{O}(-n-1) \cong \bigwedge^n \Omega_{X/R}$. In particular, we obtain isomorphisms $H^{n-1}(X, \mathscr{L}^\vee \otimes \omega_X) \cong H^i(X, \mathscr{L})^\vee$.

The goal of this section is to understand how this generalizes. We will (mostly) content ourselves with understanding the situation for a proper (or even projective) k-scheme X (where k is some field).

3.1. The abstract approach.

Using the machinery of derived categories and a suitable version of the Brown representability theorem for triangulated categories, Neeman has proved that for a morphism $f\colon X \to S$ of noetherian (or more generally: qcqs) schemes, the derived pushforward functor $Rf_*\colon D_{\mathrm{qcoh}}(X) \to D_{\mathrm{qcoh}}(S)$ admits a right adjoint $f^\times$. For $X \to \mathrm{Spec}(k)$ proper, we then call $\omega_X^\bullet := f^\times \mathscr{O}_{\mathrm{Spec}\, k} \in D_{\mathrm{qcoh}}(X)$ the *dualizing complex* of X. Since $f^\times$ is by definition right adjoint to Rf_*, for every $F \in D_{\mathrm{qcoh}}(X)$ (and in particular for every quasi-coherent $\mathscr{O}_X$-module F) we obtain the following very general form of Grothendieck-Serre duality,

$$H^i(X; F)^\vee = \mathrm{Hom}_{D(k)}(Rf_*F[i], k) = \mathrm{Hom}_{D(X)}(F[i], \omega_X^\bullet) = \mathrm{Ext}^{-i}_{\mathscr{O}_X}(F, \omega_X^\bullet).$$

The formula simplifies for example if F is a locally free $\mathscr{O}_X$-module (because then $\mathrm{Ext}^{-i}_{\mathscr{O}_X}(F, \omega_X^\bullet) = H^{-i}(X, F^\vee \otimes_{\mathscr{O}_X} \omega_X^\bullet))$ and especially if the complex $\omega_X^\bullet$ is concentrated in a single degree. This is the case if X is smooth over k, in which case $\omega_X^\bullet = \left(\bigwedge^{\dim X} \Omega_{X/k}\right)[\dim X]$.

Now consider a closed immersion $i\colon X \to Y$ of S-schemes (where we again assume that all schemes are noetherian). See [GW2] Section (25.8). To describe the functor $i^\times$, we start with the following elementary result.

Lemma 3.1. *Let $\varphi\colon A \to B$ be a ring homomorphism, let M be an A-module and let N be a B-module. Then $\mathrm{Hom}_A(B, M)$ is a B-module in a natural way, which we denote by $\mathrm{Hom}_A^B(B, M)$. We have identifications*

$$\mathrm{Hom}_A(N, M) = \mathrm{Hom}_B(N, \mathrm{Hom}_A^B(B, M))$$

functorial in M and N. (Here on the left we consider N as an A-module via φ.

Globalizing this, for a closed immersion $i\colon Z \to X$ of schemes and an $\mathscr{O}_X$-module $\mathscr{F}$, we write $\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}_Z, \mathscr{F}) := i^* \mathscr{H}om_{\mathscr{O}_X}(i_*\mathscr{O}_Z, \mathscr{F})$.

Lemma 3.2. *Let $i\colon Z \to X$ be a closed immersion of schemes. The functor $\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}, -)$ from the category of $\mathscr{O}_X$-modules to the category of $\mathscr{O}_Z$-modules is right adjoint to the direct image functor i_*.*

The lemma "formally" implies an analogous adjunction between the derived functors $R\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}, -)$ and Li_*. Since i_* is exact, we can identify $Li_* = i_* = Ri_*$. One checks that $i_* R\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}_Z, \mathscr{F}) = R\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_Z, \mathscr{F})$, i.e., when considered as an $\mathscr{O}_X$-module, this is just the usual $R\mathscr{H}om$ functor.

Note however that we do not immediately get a description of $i^\times$ because $i^\times$ is the right adjoint of $Ri_*\colon D_{\text{qcoh}}(Z) \to D_{\text{qcoh}}(X)$, and in general an object of the form $R\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}_Z, \mathscr{F})$ will not lie in $D_{\text{qcoh}}(Z)$ (cf. Example **??** below). This is however true under an additional assumption, namely for F in $D^+_{\text{qcoh}}(X)$ (or in $D^+_{\text{coh}}(X)$) the complex $R\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}_Z, F)$ lies in $D^+_{\text{qcoh}}(Z)$ (or in $D^+_{\text{coh}}(Z)$, respectively). Therefore we have

Proposition 3.3. *Let $i\colon Z \to X$ be a closed immersion of noetherian schemes. Let F in $D^+_{\text{qcoh}}(X)$. Then $i^\times F = R\mathscr{H}om_{\mathscr{O}_X}^{\mathscr{O}_Z}(\mathscr{O}_Z, F)$.*

This gives us a strategy of constructing a dualizing complex/"dualizing sheaf" for closed subschemes of projective space. Formally, we will not use any of the results above; they only serve as a motivation/explanation of why the definitions below are sensible.

3.2. **The dualizing sheaf of a projective scheme 1.**

We start by slightly generalizing the statement of Serre duality for projective space, as follows. Nov. 22, 2023

Proposition 3.4. *Let k be a field, $n \geq 1$, $X = \mathbb{P}^n_k$. Write $\omega_X = \bigwedge^n \Omega_{X/k} \cong \mathscr{O}_X(-n-1)$.*

(1) *We have $H^n(X, \omega_X) \cong k$ (and we fix one such isomorphism).*

(2) *For every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the natural pairing*

$$\operatorname{Hom}(\mathscr{F}, \omega_X) \times H^n(X, \mathscr{F}) \to H^n(X, \omega_X) \cong k$$

is perfect, i.e., it induces an isomorphism $\operatorname{Hom}(\mathscr{F}, \omega_X) \cong H^n(X, \mathscr{F})^\vee$.

(3) *For every $i \geq 0$, we have an isomorphism*

$$\operatorname{Ext}^i(\mathscr{F}, \omega_X) \cong H^{n-i}(X, \mathscr{F})^\vee.$$

(See below for a brief reminder on the Ext functor.)

Proof. We have already seen Part (1), as well as Part (2) for $\mathscr{F}$ a line bundle. Clearly, then (2) also holds for finite direct sums of line bundles. For a general $\mathscr{F}$, we can find a presentation

$$\mathscr{E}' \to \mathscr{E} \to \mathscr{F} \to 0$$

where $\mathscr{E}$ and $\mathscr{E}$ are sums of line bundles and the sequence is exact. Since the functors $\operatorname{Hom}(-, \omega_X)$ and $H^n(X, -)^\vee$ are left exact and the result holds for $\mathscr{E}$ and $\mathscr{E}'$, it follows for $\mathscr{F}$.

In view of Part (2), Part (3) follows if we can show that the δ-functors $(\mathrm{Ext}^i(-,\omega_X))_i$ and $(H^{n-i}(X,-)^\vee))_i$ are universal. To prove this, it is enough to show they are coeffaceable. But any $\mathscr{F}$ can be written as a quotient of an $\mathscr{O}_X$-module of the form $\mathscr{O}_X(-d)^{\oplus N}$ with $d \gg 0$, and both functors vanish on such sheaves for $i > 0$. (In fact, $d > n+1$ is enough since then $\omega_X(d)$ has no higher cohomology.) □

We use Parts (1) and (2) of the previous proposition to define the notion of *dualizing sheaf* for an arbitrary proper k-scheme X. (This already characterizes a dualizing sheaf, and in fact we will see below that Part (3) will not hold in general; this is related to the fact that the dualizing sheaf captures only one cohomology object of the dualizing complex of the previous section, so unless that complex is concentrated in a single degree, the dualizing sheaf will not capture the full duality.)

Definition 3.5. *Let k be a field and let X be a proper k-scheme of dimension n. A coherent $\mathscr{O}_X$-module ω_X such that there exist isomorphisms*

$$\mathrm{Hom}(\mathscr{F},\omega_X) = H^n(X,\mathscr{F})^\vee,$$

functorial on $\mathscr{F}$, is called a dualizing sheaf *on X. In other words, a dualizing sheaf is a coherent $\mathscr{O}_X$-module ω_X, together with a homomorphism $H^n(X,\omega_X) \to k$ (the element of $H^n(X,\omega_X)^\vee$ corresponding to id_{ω_X}, called the* trace map*), if for every coherent $\mathscr{O}_X$-module the natural pairing*

$$\mathrm{Hom}(\mathscr{F},\omega_X) \times H^n(X,F) \to H^n(X,\omega_X) \xrightarrow{t} k$$

is perfect, i.e., induces an isomorphism $\mathrm{Hom}(\mathscr{F},\omega_X) \cong H^n(X,\omega_X)^\vee$.

Since a dualizing sheaf is defined as the object representing a certain functor, it is clear that it is unique up to unique isomorphism (if it exists).

By the above, $\bigwedge^n \Omega_{\mathbb{P}^n_k/k}$ is a dualizing sheaf on projective space $\mathbb{P}^n_k$. We will see below that more generally for every *smooth* projective k-scheme X of dimension n, $\bigwedge^n \Omega_{X/k}$ is a dualizing sheaf on X.

Theorem 3.6. *For every proper scheme over a field, a dualizing sheaf exists.*

We will not prove the theorem here, but will rather concentrate on the case of projective schemes. (In terms of the abstract theory, writing $\omega_X^\bullet$ for the dualizing complex of X defined above, one can show that $H^i(\omega_X^\bullet) = 0$ for all $i \notin [-\dim(X), 0]$; it follows that $H^{-\dim(X)}(\omega_X^\bullet)$ is a dualizing sheaf on X.)

3.3. $\mathscr{E}xt$ **sheaves.** By and large we follow [H] III.6. See also [GW2] Sections (F.52), (21.18), (21.21), (22.17) for more general results in the context of derived categories.

Definition 3.7.

(1) *Let $\mathcal{A}$ be an abelian category with enough injectives. We define the* Ext *functor from $\mathcal{A}$ to the category of abelian groups (for $\mathscr{F}$ in $\mathcal{A}$) as* $\operatorname{Ext}^i(\mathscr{F}, -) = R^i \operatorname{Hom}(\mathscr{F}, -)$. *(One can show that*

$$\operatorname{Ext}^i(\mathscr{F}, \mathscr{G}) = \operatorname{Hom}_{D(\mathcal{A})}(\mathscr{F}, \mathscr{G}[i]).$$

This identity can be used as a definition of Ext *groups for arbitrary abelian categories.)*

(2) *Now let X be a ringed space. For an $\mathscr{O}_X$-module $\mathscr{F}$ we define the $\mathscr{E}xt$ sheaf functor from ($\mathscr{O}_X$-Mod) to ($\mathscr{O}_X$-Mod) by $\mathscr{E}xt^i(\mathscr{F}, -) = R^i \mathscr{H}om(\mathscr{F}, -)$.*

More explicitly, the definition means that we can compute $\operatorname{Ext}^i(\mathscr{F}, \mathscr{G})$ using an injective resolution of $\mathscr{G}$.

Proposition 3.8. *Let X be a ringed space and let $\mathscr{F}$, $\mathscr{G}$ be $\mathscr{O}_X$-modules.*

$$\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_{|U} = \mathscr{E}xt^i_{\mathscr{O}_U}(\mathscr{F}_{|U}, \mathscr{G}_{|U}).$$

Proof. This follows from the following fact about injective $\mathscr{O}_X$-modules: If $\mathscr{I}$ is an injective $\mathscr{O}_X$-module, then the restriction $\mathscr{I}_{|U}$ is an injective $\mathscr{O}_U$-module. (In fact, the restriction functor admits an exact left adjoint functor, the extension by zero functor, and hence preserves the class of injective objects.) □

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Since $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, -)$ is the identity functor, we have $\mathscr{E}xt^i(\mathscr{O}_X, \mathscr{G}) = \mathscr{G}$ if $i = 0$, and $= 0$ if $i > 0$. Similarly, we have $\operatorname{Ext}^i(\mathscr{O}_X, \mathscr{G}) = H^i(X, \mathscr{G})$ (where we use that H^i which we defined as the derived functor of $\Gamma\colon \mathrm{Ab}_X \to \mathrm{Ab}$ restricts to the derived functor of $\Gamma\colon (\mathscr{O}_X\text{-Mod} \to \mathrm{Ab})$.

Since the functor $\operatorname{Hom}(-, \mathscr{I})$ for an injective $\mathscr{O}_X$-module $\mathscr{I}$ is exact, for an $\mathscr{O}_X$-module $\mathscr{G}$ the family $(\operatorname{Ext}^i(-, \mathscr{G}))_i$ is a δ-functor (in particular, we obtain a long exact cohomology sequence when we plug in a short exact sequence in the first entry). A similar remark holds for $\mathscr{E}xt$ sheaves.

Similarly, one shows that $\operatorname{Ext}^i(\mathscr{F}, \mathscr{G})$ and $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ can be computed using a projective resolution of $\mathscr{F}$. This is useful working in the derived category of modules over a ring. But when working with $\mathscr{O}_X$-modules for a ringed space X (even a noetherian scheme) in general projective objects are very rare. Note that even the structure sheaf $\mathscr{O}_X$ is not a projective $\mathscr{O}_X$-module in general.

Therefore it is useful to study whether other types of resolutions (specifically, a resolution by locally free $\mathscr{O}_X$-modules of finite rank) can be used for computing $\mathscr{E}xt$ sheaves.

Proposition 3.9. *The $\mathscr{E}xt$ sheaves can be computed using a locally free finite rank resolution for the first entry, i.e.,*

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = H^i(\mathscr{H}om(\mathscr{E}_\bullet, \mathscr{G})),$$

if $\cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$ is exact and all $\mathscr{E}_i$ are locally free of finite rank.

Proof. Both sides are δ-functors in $\mathscr{G}$, and they agree for $i = 0$. Both sides vanish for $i > 0$ and $\mathscr{G}$ injective (for the left hand side, use that $\mathscr{H}om(-, \mathscr{G})$ is exact since injectivity is preserved under restriction to opens, cf. the proof of Proposition 3.8) and hence are universal δ-functors. □

Question. Do you see why for the Ext groups this cannot possibly hold? Asked differently, what goes wrong in the proof, if we replace $\mathscr{E}xt$ by Ext and $\mathscr{H}om$ by Hom? (*Hint.* Where have we used that the $\mathscr{E}_i$ are locally free?)

Corollary 3.10. *Let X be a noetherian scheme and let $\mathscr{F}$, $\mathscr{G}$ be coherent $\mathscr{O}_X$-modules. Then for all $i \geq 0$, the $\mathscr{O}_X$-module $\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is coherent.*

Corollary 3.11. *Let X be a noetherian scheme, let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules and assume that $\mathscr{F}$ is coherent. Then for all $x \in X$ we have*

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})_x = \mathrm{Ext}^i_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).$$

Proof. By Proposition 3.8 we can reduce to the case that $X = \operatorname{Spec} A$ is affine. Choose a resolution $\mathscr{P}_\bullet \to \mathscr{F} \to 0$ of $\mathscr{F}$ by free $\mathscr{O}_X$-modules $\mathscr{P}_i$. Passing to the stalks at x, we obtain a free resolution of $\mathscr{F}_x$ as an $\mathscr{O}_{X,x}$-module.

By the above, $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = \mathscr{H}om(\mathscr{P}_i, \mathscr{G})$, and $\mathscr{H}om(\mathscr{P}_i, \mathscr{G})_x = \mathrm{Hom}(\mathscr{P}_{i,x}, \mathscr{G}_x)$ since $\mathscr{P}_i$ is of finite presentation (this is enough since the result clearly holds for $\mathscr{O}_X$ in the first entry, thus for finite free modules, and by the five lemma follows for modules of finite presentation). □

Example 3.12. Let R be a discrete valuation ring with field of fractions K, $X = \operatorname{Spec} R$, $\eta \in X$ the generic point. Then

$$\mathscr{H}om_{\mathscr{O}_X}(\bigoplus_{\mathbb{N}} R, R)_\eta = \left(\prod_{\mathbb{N}} R\right) \otimes_R K \neq \prod_{\mathbb{N}} K = \mathrm{Hom}_K(K \otimes_R \bigoplus_{\mathbb{N}} R, K).$$

Proposition 3.13. *Let X be a ringed space, let $\mathscr{F}$, $\mathscr{G}$ be $\mathscr{O}_X$-modules, and $\mathscr{E}$ a locally free $\mathscr{O}_X$-modules. We write $\mathscr{E}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X)$. Then for all $i \geq 0$ we have natural identifications*

$$\mathrm{Ext}^i(\mathscr{F} \otimes \mathscr{E}, \mathscr{G}) = \mathrm{Ext}^i(\mathscr{F}, \mathscr{G} \otimes \mathscr{E}^\vee)$$

and

$$\mathscr{E}xt^i(\mathscr{F} \otimes \mathscr{E}, \mathscr{G}) = \mathscr{E}xt^i(\mathscr{F}, \mathscr{G} \otimes \mathscr{E}^\vee) = \mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) \otimes \mathscr{E}^\vee.$$

Proof. All the above functors are δ-functors in $\mathscr{G}$ which are moreover effaceable and hence universal. Therefore it is enough to show the equalities in the case $i = 0$ where they are clear. □

Excursion: Spectral sequences 3.14. See [GW2] Sections (F.21) – (F.25), (F.50). Other references are [We], [Gr], [EGA] III_1 §11.

Proposition 3.15. (Leray spectral sequence)

(1) *Let $f\colon X \to Y$ be a morphism of ringed spaces and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. There is a convergent spectral sequence*
$$E_2^{pq} = H^p(Y, R^q f_* \mathscr{F}) \Longrightarrow H^{p+q}(X, \mathscr{F}).$$

(2) *Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms of ringed spaces. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. There is a convergent spectral sequence*
$$E_2^{pq} = R^p g_*(R^q f_* \mathscr{F}) \Longrightarrow R^{p+q}(g \circ f)_* \mathscr{F}.$$

Proposition 3.16. (Čech vs. derived functor cohomology) *Let X be a ringed space and let $\mathscr{U}$ be an open cover of X. There is a convergent spectral sequence*
$$E_2^{pq} = \check{H}^p(\mathscr{U}, \mathscr{H}^q(\mathscr{F})) \Longrightarrow H^{p+q}(X, \mathscr{F}).$$
Here $\mathscr{H}^q(\mathscr{F})$ denotes the presheaf $U \mapsto H^q(U, \mathscr{F})$.

Proof. This is the Grothendieck spectral sequence for the composition of functors $\check{H}^0$ from the category of presheaves(!) on X to the category of abelian groups, and the inclusion ι of the category of $\mathscr{O}_X$-modules into the category of presheaves of $\mathscr{O}_X$-modules. Note that ι preserves the property of being injective because it has an exact left adjoint functor (namely sheafification). The composition is just the global section functor on the category of sheaves. One concludes by noting that $\mathscr{H}^q$ is the q-th right derived functor of ι. $\square$

Proposition 3.17. (The local-to-global spectral sequence for Ext)

(1) *Let X be a ringed space and let $\mathscr{F}$, $\mathscr{G}$ be $\mathscr{O}_X$-modules. We have a convergent spectral sequence*
$$E_2^{pq} = H^p(X, \mathscr{E}xt^q(\mathscr{F}, \mathscr{G})) \Longrightarrow \operatorname{Ext}^{p+q}(\mathscr{F}, \mathscr{G}).$$

(2) *Let $i\colon Z \to X$ be a closed immersion of schemes (or of arbitrary ringed spaces), let $\mathscr{F}$ be an $\mathscr{O}_Z$-module and $\mathscr{G}$ an $\mathscr{O}_X$-module. We have a convergent spectral sequence*
$$E_2^{pq} = \operatorname{Ext}^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}xt^q_{\mathscr{O}_Z}(\mathscr{O}_Z, \mathscr{G})) \Longrightarrow \operatorname{Ext}^{p+q}_{\mathscr{O}_X}(i_* \mathscr{F}, \mathscr{G}).$$

Corollary 3.18. *Let k be a field, $i\colon X \to \mathbb{P}^n_k$ a closed immersion and $\mathscr{O}_X(1) := i^* \mathscr{O}_{\mathbb{P}^n_k}(1)$. Let $\mathscr{F}, \mathscr{G}$ be coherent $\mathscr{O}_X$-modules. Then for all $i \geq 0$ and all $q \gg 0$ (depending on $\mathscr{F}, \mathscr{G}$) we have*
$$\operatorname{Ext}^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(q)) = \Gamma(X, \mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(q))).$$

3.4. **The dualizing sheaf of a projective scheme 2.** With the results on $\mathscr{E}xt$ sheaves, we can construct a dualizing sheaf for projective schemes over a field.

Theorem 3.19. *Let k be a field, $n \geq 1$ and let $i\colon X \to \mathbb{P}^n_k$ be a closed immersion. Let $r = n - \dim X$ be the codimension of X. Let $\omega = \omega_{\mathbb{P}^n_k}$ Then*

(1) $\mathscr{E}xt^i_{\mathscr{O}_{\mathbb{P}^n_k}}(\mathscr{O}_X, \omega) = 0$ *for all* $i < r$, *and*
(2) $\omega_X = \mathscr{E}xt^r_{\mathscr{O}_{\mathbb{P}^n_k}}(\mathscr{O}_X, \omega)$ *is a dualizing sheaf on* X.

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