

ALGEBRAIC GEOMETRY 2, SUMMER TERM 2026. LECTURE COURSE NOTES.

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INTRODUCTION

This lecture course is a continuation of the course *Algebraic Geometry 1* which covered the definition of schemes, and some basic notions about schemes and scheme morphisms, such as immersions and subschemes, fiber products of schemes and separated morphisms.

The main object of study of this term's course will be the notion of $\mathcal{O}_X$ -module, a natural analogue of the notion of module over a ring in the context of sheaves of rings. As we will see, the $\mathcal{O}_X$ -modules on a scheme X contain a lot of information about the geometry of this scheme, and we will study them using a variety of methods. In particular, we will introduce the notion of *cohomology groups*, a powerful algebraic tool that makes its appearance in many areas of algebra and geometry.

These notes are based on the notes for similar courses which I taught in 2023 and 2019, and will be updated or modified to reflect the content of the current class while the term proceeds.

References. The books [GW1], [GW2] by Wedhorn and myself, by Hartshorne [H], and by Mumford [Mu]. More precise references are given in most of the individual sections. Mumford still uses the ancient terminology and calls a *prescheme* what we call a scheme, and a *scheme* what we call a separated scheme. Further references: [Stacks], [EGA].

We give a brief overview of some of the topics we will discuss in this class.

Proper schemes and morphisms. (Section (1.6)) Similarly as separatedness is the algebro-geometric version of the Hausdorff property, properness is the algebro-geometric version of compactness; we will see that - as expected, cf. the Riemann sphere - projective space and all of its closed subschemes are proper. (The converse is, interestingly, not true: There exist proper schemes (over a field, say) which cannot be embedded into any projective space.)) April 14,
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Divisors and line bundles. (Chapter 3) Assume we have a closed subscheme $X \subseteq \mathbb{P}_k^n$. We can intersect X with the "coordinate hyperplanes" $V_+(X_i)$ to obtain subschemes of X of a very special form, so-called divisors.

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We will discuss several perspectives on the notion of divisor. It is an interesting question whether, given a divisor, it arises in the way described above.

$\mathcal{O}_X$ -Modules. (Chapter 2) Similarly to the notion of module over a ring, there is a notion of sheaves of modules over a sheaf of rings which we can apply in particular to the structure sheaf of a scheme or of any ringed space. Such $\mathcal{O}_X$ -modules contain a lot of interesting information about the underlying space X . There is a close connection to the notion of divisor which we will discuss in the course. (And for this reason, in the course we will first talk about $\mathcal{O}_X$ -modules, and then about divisors.)

Cohomology of sheaves. (Chapter 6) We know that a surjective sheaf morphism might not induce surjections on the sets of sections over a fixed open of the underlying space (for simplicity take the whole space X). So the functor $\Gamma(X, -)$ taking global sections (on the category of sheaves of abelian groups say, where we have a notion of exact sequences) is not exact. Studying in which way exactness fails leads, by the general formalism of "derived functors", to the notion of cohomology groups $H^i(X, \mathcal{F})$ on X with coefficients in a sheaf $\mathcal{F}$ of abelian groups. (Often $\mathcal{F}$ will be a $\mathcal{O}_X$ -module). This is an extremely powerful tool to "algebraize" geometric information. After this "controlled loss of information" often things are easier to work with.

The Theorem of Riemann-Roch. (Theorem 3.45, Section (6.17)) This famous theorem is a specific example where cohomology of $\mathcal{O}_X$ -modules attached to divisors on a smooth projective curve can be used very profitably in order to study the geometry of the curve. Among many other things it yields a very clean description of the group structure on an elliptic curve, giving us the associativity (which we had to leave open in Part 1) basically for free.

1. PROPER MORPHISMS

The functorial point of view.**(1.1) Schemes as functors.**

References: [GW1] Sections (4.1), (4.2); [Mu] II.6.

As we have discussed in Algebraic Geometry 1, to a scheme X we can attach its functor of T -valued points:

$$h_X: (\text{Sch})^{\text{opp}} \rightarrow (\text{Sets}), \quad T \mapsto X(T) := \text{Hom}(T, X),$$

which on morphisms is just given by composition: $f: T' \rightarrow T$ is mapped under h_X to the map $X(T) \rightarrow X(T')$, $\alpha \mapsto \alpha \circ f$.

Example 1.1.

- (1) For any affine scheme $X = \text{Spec } A$, we have $X(T) = \text{Hom}(A, \Gamma(T, \mathcal{O}_T))$. For example, this gives $\mathbb{A}_R^n(T) = \Gamma(T, \mathcal{O}_T)^n$ for any ring R and any R -scheme T (where we understand $\mathbb{A}_R^n(T)$ as the set of morphisms $T \rightarrow \mathbb{A}_R^n$ of R -schemes).
- (2) It is more difficult to describe $\mathbb{P}^n(T)$ for general T (for T the spectrum of a field, we have the description by homogeneous coordinates). We will come back to this later.

Using the notion of a morphism of functors, we can speak of the category $\widehat{\mathcal{C}} := \text{Func}((\text{Sch})^{\text{opp}}, (\text{Sets}))$ of all such functors, and we obtain a functor $h: (\text{Sch}) \rightarrow \widehat{\mathcal{C}}$, $X \mapsto h_X$, which on morphisms is – once again – defined by composition: $\alpha: X' \rightarrow X$ is mapped by h to the morphism $h_{X'} \rightarrow h_X$ of functors given by $X'(T) \rightarrow X(T)$, $\beta \mapsto \beta \circ \alpha$.

Even though at first sight this may look complicated, this is an entirely “formal” (i.e., category-theoretic) procedure which has nothing to do with schemes. In fact, if $\mathcal{C}$ is any category (as always, assumed to be *locally small*, i.e., each $\text{Hom}(X, Y)$ is a set), for an object X of $\mathcal{C}$ we can define the functor

$$h_X: \mathcal{C}^{\text{opp}} \rightarrow (\text{Sets}), \quad T \mapsto X(T) := \text{Hom}_{\mathcal{C}}(T, X),$$

and now setting $\widehat{\mathcal{C}} := \text{Func}(\mathcal{C}^{\text{opp}}, (\text{Sets}))$, we obtain a functor $h: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$.

Theorem 1.2. (Yoneda Lemma) *For any category $\mathcal{C}$, the functor h constructed above is fully faithful.*

Proof. Given X and Y and a morphism $\Phi: h_X \rightarrow h_Y$, we obtain a morphism $X \rightarrow Y$ by applying Φ to $\text{id}_X \in h_X(X)$. One checks that this is an inverse of the map $\text{Hom}(X, Y) \rightarrow \text{Hom}(h_X, h_Y)$ given by h . $\square$

Remark 1.3. The Yoneda lemma says that the map $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}(h_X, h_Y)$ is a bijection for all X, Y . We may rewrite the left hand side as $h_Y(X)$. More generally, with basically the same argument one can

show that for every functor $F: \mathcal{C}^{\text{opp}} \rightarrow (\text{Sets})$ there are identifications $F(X) = \text{Hom}_{\text{Func}(\mathcal{C}^{\text{opp}}, (\text{Sets}))}(h_X, F)$ that are functorial in X .

We will mostly apply the Yoneda Lemma to the category of schemes, or the category of S -schemes for some fixed scheme S . Let us list some of its consequences (as before, these facts are not specific to the category of schemes):

(1) Let X, Y be schemes. The following are equivalent:

- (i) $X \cong Y$,
- (ii) $h_X \cong h_Y$ (isomorphism of functors)
- (iii) there exists a family $f_T: X(T) \rightarrow Y(T)$ of bijections of sets that is functorial in T , i.e., for every scheme morphism $T' \rightarrow T$, the diagram

$$\begin{array}{ccc} X(T) & \xrightarrow{f_T} & Y(T) \\ \downarrow & & \downarrow \\ X(T') & \xrightarrow{f_{T'}} & Y(T') \end{array}$$

is commutative.

- (2) Let X, Y be schemes. Giving a scheme morphism $X \rightarrow Y$ is equivalent to giving a family of maps $f_T: X(T) \rightarrow Y(T)$ of sets for each scheme T , that is functorial in T (same condition as in (1) (iii)). *Example.* The determinant of a matrix is a scheme morphism $\mathbb{A}^{n^2} \rightarrow \mathbb{A}^1$.
- (3) A diagram of scheme morphisms is commutative if and only if for every scheme T the diagram (in the category of sets) obtained by replacing each scheme by its set of T -valued points, and replacing the scheme morphisms by the induced maps of sets, is commutative.

Given a functor $F: \mathcal{C}^{\text{opp}} \rightarrow (\text{Sets})$, it “usually” will not be isomorphic to a functor of the form h_X . If it is, it is called *representable* (by X), and this is a very special property. On the other hand, especially in algebraic geometry, i.e., when $\mathcal{C}$ is the category of schemes (or of S -schemes for some fixed scheme S), it turns out that many naturally appearing functors are in fact representable. Proving such representability results is often quite difficult, but at the same time extremely useful, and this approach yields some of the most interesting examples of schemes.

Fiber products and base change.

(1.2) Recap: fiber products of schemes.

References: [GW1] Sections (4.4)–(4.6); [H] II.3; [Mu] II.2.

The universal property of a *fiber product* generalizes the universal property of a product (of two objects, in any category). It is defined as follows. (See the lecture notes for Algebraic Geometry 1 for a bit more background, e.g., an explanation of the term *fiber product*.)

Definition 1.4. Let $\mathcal{C}$ be a category. We fix morphisms $f: X \rightarrow S$ and $g: Y \rightarrow S$. Then an object P in $\mathcal{C}$ together with morphisms $p: P \rightarrow X$ and $q: P \rightarrow Y$ such that $f \circ p = g \circ q$ is called a fiber product of X and Y over S , if for every object T of $\mathcal{C}$ together with morphisms $\alpha: T \rightarrow X$ and $\beta: T \rightarrow Y$ with $f \circ \alpha = g \circ \beta$, there exists a unique morphism $\xi: T \rightarrow P$ with $p \circ \xi = \alpha$, $q \circ \xi = \beta$.

We say that a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is *cartesian*, if it is a fiber product diagram, i.e., if A satisfies the universal property defining the fiber product of B and C over D .

Theorem 1.5.

- (1) If $f: X \rightarrow S$, $g: Y \rightarrow S$ are morphisms of schemes, then the fiber product of X and Y over S exists.
- (2) If in (1) $X = \text{Spec } A$, $Y = \text{Spec } B$, $S = \text{Spec } R$ are affine schemes (so that f and g are given by ring homomorphisms $R \rightarrow A$, $R \rightarrow B$), then $\text{Spec } A \otimes_R B$ together with the morphisms induced by the natural maps $A \rightarrow A \otimes_R B$, $B \rightarrow A \otimes_R B$, is the fiber product of X and Y over S .
- (3) For f , g as in (1), and open covers $S = \bigcup_i U_i$, $f^{-1}(U_i) = \bigcup_j V_{ij}$, $g^{-1}(U_i) = \bigcup_k W_{ik}$, for all i, j, k , the natural morphism $V_{ij} \times_{U_i} W_{ik} \rightarrow X \times_S Y$ induced by the universal property of the fiber product is an open immersion, and taken together the open subschemes of the above form cover $X \times_S Y$.

Example 1.6. If $f: X \rightarrow S$ is any morphism and $V \rightarrow S$ is an open immersion, then we can identify $X \times_S V = f^{-1}(V)$ as X -schemes, i.e., the projection $X \times_S V \rightarrow X$ is an open immersion which induces an isomorphism $X \times_S V \cong f^{-1}(V)$ (where we view $f^{-1}(V)$ as an open subscheme of X).

Example 1.7. (Fibers of a morphism) If $f: X \rightarrow S$ is a morphism of schemes and $s \in S$, then we call the fiber product $X_s := f^{-1}(s) := X \times_S \text{Spec } \kappa(s)$ (with respect to f and the natural morphism $\text{Spec } \kappa(s) \rightarrow S$) the *scheme-theoretic fiber of f over s* . The projection $X_s \rightarrow X$ induces a homeomorphism between the topological space of the scheme X_s and the fiber of the continuous map f over s . Cf. the Algebraic Geometry 1 class, or see [GW1] Section (4.8).

Lemma 1.8. In the list below, the “obvious” maps between fiber products are isomorphisms:

- (1) $X \times_S S \cong X$,
- (2) $X \times_S Y \cong Y \times_S X$,
- (3) $(X \times_S Y) \times_T Z \cong X \times_S (Y \times_T Z)$ (and this allows us to omit the parentheses in expressions like these).

- (4) Let $f: X \rightarrow Y$ be a morphism and let $\Gamma_f: X \xrightarrow{(\text{id}, f)} X \times_S Y$ be the graph of f . Then the following diagram is a fiber product diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \Gamma_f & & \downarrow \Delta \\ X \times_S Y & \xrightarrow{f \times \text{id}} & Y \times_S Y. \end{array}$$

(These claims have to be read in the “natural way”, e.g., in (1) the fiber product on the left is formed with respect to $\text{id}: S \rightarrow S$, and in (3) morphisms $X \rightarrow S$, $Y \rightarrow S$, $Y \rightarrow T$, $Z \rightarrow T$ are given and used to form the fiber products, and the morphism $X \times_S Y \rightarrow T$ is the composition $X \times_S Y \rightarrow Y \rightarrow T$ of the projection to the second factor with the given morphism $Y \rightarrow T$, and similarly on the right hand side.)

Proof. These properties can easily be checked using the universal property (or, what more or less amounts to the same, by the Yoneda lemma). In any case, this reduces to checking the above claims for fiber products of sets, where they follow immediately from the explicit description of fiber products of sets. $\square$

Example 1.9. (Group schemes) Let S be a scheme. A *group scheme over S* is an S -scheme G together with a functor $h: ((\text{Sch})/S)^{\text{opp}} \rightarrow (\text{Grp})$, such that h_G is the composition of h and the forgetful functor $(\text{Grp}) \rightarrow (\text{Sets})$. In other words, for every S -scheme T , we are given a group structure on $G(T)$, and for every morphism $T' \rightarrow T$, the induced map $G(T) \rightarrow G(T')$ is a group homomorphism.

In view of the above discussion, we can express this structure equivalently by giving a multiplication morphism $m: G \times_S G \rightarrow G$, a morphism $i: G \rightarrow G$ (“inverse element”) that induces the map $g \mapsto g^{-1}$ on each $G(T)$, and a morphism $S \rightarrow G$ (“neutral element”) that induces the neutral element in each $G(T)$ (note that for every S -scheme T , the set $S(T)$ is a singleton). The morphisms m, i, e have to satisfy certain conditions reflecting the group axioms; the conditions can be expressed by requiring that certain diagrams be commutative. See [GW1] Section (4.15).

Some examples: Fix an affine scheme $S = \text{Spec}(R)$. We will consider group schemes over S .

- (1) The trivial group scheme $G = S$. Then $G(T)$ is a singleton set for every S -scheme T , and we give it the structure of the trivial group.
- (2) The *additive group* $\mathbb{G}_a$ (over S). As a scheme, we define it as $\mathbb{G}_a = \text{Spec } R[X]$. Then for every S -scheme T we have, since $\mathbb{G}_a$ is affine,

$$\mathbb{G}_a(T) = \text{Hom}_S(T, \mathbb{G}_a) = \text{Hom}_R(R[X], \Gamma(T, \mathcal{O}_T)) = \Gamma(T, \mathcal{O}_T)$$

and we equip it with the group structure given by the addition of the ring $\Gamma(T, \mathcal{O}_T)$.

- (3) The *multiplicative group* $\mathbb{G}_m$ (over S). As a scheme, we define it as $\mathbb{G}_m = \operatorname{Spec} R[X, X^{-1}]$. Then for every S -scheme T we have, since $\mathbb{G}_m$ is affine,

$$\mathbb{G}_m(T) = \operatorname{Hom}_S(T, \mathbb{G}_m) = \operatorname{Hom}_R(R[X, X^{-1}], \Gamma(T, \mathcal{O}_T)) = \Gamma(T, \mathcal{O}_T)^\times$$

and we equip it with the multiplicative group structure inherited from the multiplication of the ring $\Gamma(T, \mathcal{O}_T)$.

- (4) The *general linear group* GL_n (as a group scheme over S). This should be a group scheme GL_n such that for every S -scheme T we have

$$GL_n(T) = GL_n(\Gamma(T, \mathcal{O}_T)),$$

where the right hand side is the “usual” general linear group of invertible matrices with entries in the ring $\Gamma(T, \mathcal{O}_T)$. Clearly this is a functorial group structure on all these sets, but we still have to justify that there exists an S -scheme that has these sets as its T -valued points. But this is not difficult: We view the affine space $\mathbb{A}_S^{n^2} = \operatorname{Spec}(R[X_{i,j}; i, j = 1, \dots, n])$ of dimension n^2 as the space of $n \times n$ -matrices, i.e.,

$$\mathbb{A}_S^{n^2}(T) = \Gamma(T, \mathcal{O}_T)^{n^2} = \operatorname{Mat}_n(\Gamma(T, \mathcal{O}_T)),$$

denote by $\det \in R[X_{i,j}; i, j = 1, \dots, n]$ the determinant of the matrix $(X_{i,j})_{i,j}$ given by the indeterminates, and define $GL_n = D(\det)$, the principal open subscheme defined by $\det$, as a scheme. This has the desired T -valued points.

- (5) Elliptic curves also are an (interesting!) example, but what we have done in Part 1 of the lecture is not a full proof that they carry a group scheme structure (but could be made into one without too much work: one would have to express the group law that we have constructed on $E(k')$ for every extension field k' of the base field k in terms of explicit formulas and use those to show that the multiplication and the rule mapping each element to its inverse (=negative) are in fact given by scheme morphisms).

(1.3) Base change.

References: [GW1] Chapter 4, in particular Sections (4.7)–(4.10).

Given scheme morphisms $f: X \rightarrow S$ and $g: S' \rightarrow S$, we call the projection $X \times_S S' \rightarrow S'$ the *morphism obtained from f by base change along g* . This defines a functor from the category of S -schemes to the category of S' -schemes.

A particularly simple example is the case where $g: V \rightarrow S$ is an open immersion. In that case the base change of f is just the restriction of f to $f^{-1}(V) \rightarrow V$.

Many properties of scheme morphisms are “stable under base change” in the following sense: A property **P** of scheme morphisms is called stable

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under base change if for every morphism $f: X \rightarrow Y$ of S -schemes that has property **P** and every scheme morphism $S' \rightarrow S$, the induced morphism $X \times_S S' \rightarrow Y \times_S S'$ also has property **P**.

Given a property **P**, to check that it is stable under base change, it is enough to check that whenever $f: X \rightarrow S$ has the property, and $g: S' \rightarrow S$ is a scheme morphism, then $X \times_S S' \rightarrow S'$ also has the property. In fact, this is clearly a special case of the above definition (namely the case where $Y = S$). On the other hand, suppose this special case is true and $f: X \rightarrow Y$ is any morphism of S -schemes. Identifying $X \times_S S' = X \times_Y (Y \times_S S')$ using the rules of “computations with fiber products” (Lemma 1.8), the base change $X \times_S S' \rightarrow Y \times_S S'$ is identified with the projection $X \times_Y (Y \times_S S') \rightarrow Y \times_S S'$. Applying the special case to $X \rightarrow Y$ and the base change $Y \times_S S' \rightarrow Y$, we obtain that $X \times_S S' = X \times_Y (Y \times_S S') \rightarrow Y \times_S S'$ has property **P**.

Proposition 1.10. *The following properties of scheme morphisms are stable under base change: Being ...*

- (1) *an isomorphism,*
- (2) *an open immersion,*
- (3) *a closed immersion,*
- (4) *an immersion,*
- (5) *surjective,*
- (6) *...most of the properties of scheme morphisms that we will get to know later in the course ...*

A notable exception is the property of being injective: Can you find an example of an injective morphism $X \rightarrow S$ of schemes and a morphism $S' \rightarrow S$ such that the base change $X \times_S S' \rightarrow S'$ is not injective?

All the properties in the above list, and also being injective, are *stable under composition*, i.e., if two composable morphisms both have the property, then so does the composition.

Example 1.11.

- (1) If $R \rightarrow R'$ is a ring homomorphism, then $\mathbb{A}_R^n \otimes_R R' := \mathbb{A}_R^n \times_{\text{Spec } R} \text{Spec } R' = \mathbb{A}_{R'}^n$. In view of this we define, for an arbitrary scheme S , $\mathbb{A}_S^n := \mathbb{A}_{\mathbb{Z}}^n \times_{\text{Spec } \mathbb{Z}} S$.
- (2) If $R \rightarrow R'$ is a ring homomorphism, then $\mathbb{P}_R^n \otimes_R R' := \mathbb{P}_R^n \times_{\text{Spec } R} \text{Spec } R' = \mathbb{P}_{R'}^n$. In view of this we define, for an arbitrary scheme S , $\mathbb{P}_S^n := \mathbb{P}_{\mathbb{Z}}^n \times_{\text{Spec } \mathbb{Z}} S$.

Remark 1.12. Recall the notion of separated morphism: $Y \rightarrow S$ is called separated, if the diagonal morphism $\Delta: Y \rightarrow Y \times_S Y$ is a *closed* immersion (it is always an immersion, and the question really is whether the topological image is a closed subspace). This is a scheme-theoretic analogue of the Hausdorff property of topological spaces.

As an application of the above we obtain the following. Let $Y \rightarrow S$ be separated and let $f: X \rightarrow Y$ be any morphism. Then the graph $\Gamma_f: X \rightarrow$

$X \times_S Y$ is a closed immersion, because being a closed immersion is stable under base change and because of the cartesian diagram in Lemma 1.8 (4).

Proper morphisms.

(1.4) Proper maps between topological spaces.

References: [Bou-TG] Ch. I §10, [Stacks] Section 005M.

Most schemes that we have encountered so far (in particular, all affine schemes, projective space over any ring, subschemes $V_+(I)$ of projective space over a ring, ...) are quasi-compact. On the other hand, from a geometric point of view, e.g., the affine line (or higher-dimensional affine space) “should not be viewed” as a compact space. The notion of properness is a suitable replacement in algebraic geometry for the notion of compactness in topology/differential geometry.

Similarly as separatedness, we will define properness in terms of fiber products of schemes, starting from a characterization of quasi-compact topological spaces, given by the notion of *proper map* between continuous spaces, which we discuss below as a motivation for the definition of proper scheme morphisms. The purpose of motivation aside, the rest of this section plays no role in the course.

Note that fiber products in the category of topological spaces exist. In fact, for continuous maps $X \rightarrow S$, $Y \rightarrow S$, the set-theoretic fiber product $X \times_S Y$, equipped with the subspace topology for the inclusion $X \times_S Y \subseteq X \times Y$ (where the right hand side carries the product topology) is easily seen to satisfy the required universal property.

Recall that a continuous map $f: X \rightarrow Y$ is *closed*, if for every closed subset $C \subseteq X$, the image $f(C) \subseteq Y$ is closed.

Definition 1.13.

- (1) We call a continuous map $f: X \rightarrow Y$ *universally closed*, if for every continuous map $Z \rightarrow Y$, the “base change” of f along $Z \rightarrow Y$, i.e., the induced map $X \times_Y Z \rightarrow Z$, is closed.
- (2) We call a continuous map $f: X \rightarrow Y$ *Bourbaki-proper*, if for every topological space Z , the induced map $f \times \text{id}_Z: X \times Z \rightarrow Y \times Z$ is closed.

Proposition 1.14. *Let $f: X \rightarrow Y$ be a continuous map. The following properties are equivalent.*

- (i) f is *universally closed*,
- (ii) f is *Bourbaki-proper*,
- (iii) f is *closed* and for every $y \in Y$ the fiber $f^{-1}(y)$ is *quasi-compact*.
- (iv) f is *closed* and for every quasi-compact subset $K \subseteq Y$, the inverse image $f^{-1}(K)$ is *quasi-compact*.

See [Stacks] Theorem 005R.

(1.5) Morphisms of finite type.

References: [GW1] Sections (10.1), (10.2).

To define proper morphisms of schemes in the following section, we also need the following ingredients.

Definition 1.15. A morphism $f: X \rightarrow Y$ is called *quasi-compact*, if for every quasi-compact open $V \subseteq Y$ the inverse image $f^{-1}(V)$ is quasi-compact.

Lemma 1.16. Let $f: X \rightarrow Y$ be a morphism of schemes. The following are equivalent.

- (i) The morphism f is quasi-compact.
- (ii) For every affine open subscheme $V \subseteq Y$, the inverse image $f^{-1}(V)$ is quasi-compact.
- (iii) There exists a cover $Y = \bigcup_i V_i$ by affine open subschemes such that for every i the inverse image $f^{-1}(V_i)$ is quasi-compact.

Note that in part (iii) of the lemma it is important to consider a cover by *affine* open subschemes.

Recall that an algebra B over a ring A is called *of finite type* (or equivalently, *finitely generated*) if there exists $n \geq 0$ and a surjective A -algebra homomorphism $A[X_1, \dots, X_n] \rightarrow B$. We then also say that the ring homomorphism $A \rightarrow B$ is of finite type. For example, if R is any ring, $f \in R$, then the homomorphism $R \rightarrow R_f$ is of finite type, since $R_f \cong R[X]/(fX - 1)$. (On the other hand, if $\mathfrak{p} \subset R$ is a prime ideal, then the homomorphism $R \rightarrow R_{\mathfrak{p}}$ typically is not of finite type.) If $A \rightarrow B$ and $B \rightarrow C$ are ring homomorphisms of finite type, then the composition $A \rightarrow C$ is also of finite type.

Definition 1.17.

- (1) A morphism $f: X \rightarrow Y$ of schemes is called *locally of finite type* (or: X is called a Y -scheme *locally of finite type*, or *locally of finite type over Y*), if for every affine open subscheme $V \subseteq Y$ and every open subscheme $U \subseteq f^{-1}(V)$, the ring homomorphism $\Gamma(V, \mathcal{O}_Y) \rightarrow \Gamma(U, \mathcal{O}_X)$ induced by the restriction $U \rightarrow V$ of f makes $\Gamma(U, \mathcal{O}_X)$ a $\Gamma(V, \mathcal{O}_Y)$ -algebra of finite type.
- (2) A morphism $f: X \rightarrow Y$ of schemes is called *of finite type* (or: X is called a Y -scheme *of finite type*, or *of finite type over Y*), if f is locally of finite type and quasi-compact.

Lemma 1.18. Let $f: X \rightarrow Y$ be a morphism of schemes. The following are equivalent.

- (i) The morphism f is locally of finite type.
- (ii) There exist a cover $Y = \bigcup_i V_i$ by affine open subschemes, and for each i a cover $f^{-1}(V_i) = \bigcup_j U_{ij}$ by affine open subschemes such that for all i, j the $\Gamma(V_i, \mathcal{O}_Y)$ -algebra $\Gamma(U_{ij}, \mathcal{O}_X)$ is of finite type.

Proof. This kind of statement holds for many interesting properties of morphisms of schemes. While the proof does not use any difficult input, it is quite long; but it nicely illustrates how to work with schemes and to eventually reduce scheme-theoretic statements to commutative algebra.

Step 1. We first show that without loss of generality, we may assume $V = Y$. In fact, if $D_{V_i}(s) \subset V_i$ is a principal open subset, then $f^{-1}(D_{V_i}(s)) = \bigcup_j D_{U_{ij}}(s_j)$, where $s_j \in \Gamma(U_{ij}, \mathcal{O}_X)$ is the image of s , and the $\Gamma(D_{V_i}(s), \mathcal{O}_Y)$ -algebra $\Gamma(D_{U_{ij}}(s_j), \mathcal{O}_X)$ is of finite type. Thus replacing the V_i by suitable principal open subsets, we may assume that V is a union of some of the V_i , and can thus replace Y by V . Moreover, we may choose these principal opens in V_i so that at the same time they are *principal* open in V (Lemma 1.19 below), so we may assume in addition that each V_i is principal open in Y .

Step 2. We may replace U_{ij} by a cover by principal opens, and thus assume that U is covered by some of the U_{ij} . Moreover, as in Step 1 we may choose these principal opens in U_{ij} so that at the same time they are *principal* open in U . We may therefore assume, without loss of generality, that $U = X$ and that every U_{ij} is principal open in X .

Step 3. We now have the following situation: $X \rightarrow Y$ is a morphism of affine schemes, say corresponding to a ring homomorphism $A \rightarrow B$. We want to show that $A \rightarrow B$ is of finite type. We have a covering $Y = \bigcup V_i$ by principal open subsets (which we may assume to be finite, since Y is quasi-compact), and coverings $f^{-1}(V_i) = \bigcup_j U_{ij}$ (which likewise we may assume to be finite) by principal opens of X , such that the homomorphism $\Gamma(V_i, \mathcal{O}_Y) \rightarrow \Gamma(U_{ij}, \mathcal{O}_X)$ is of finite type. Since $\Gamma(V, \mathcal{O}_Y) \rightarrow \Gamma(V_i, \mathcal{O}_Y)$ is also of finite type and this property is stable under composition, we also have that $\Gamma(V, \mathcal{O}_Y) \rightarrow \Gamma(U_{ij}, \mathcal{O}_X)$ is of finite type. We may therefore “forget about the V_i ” and simply apply Lemma 1.20 to $A \rightarrow B$ and the cover $\text{Spec}(B) = \bigcup_{i,j} U_{ij}$. $\square$

In the proof we have used the following lemmas.

Lemma 1.19. *Let X be a scheme, $x \in X$, and U, V affine open subschemes of X which both contain x . Then there exists an open neighborhood $W \subseteq U \cap V$ of x which is principal open both in U and in V .*

Proof. One easily checks that a principal open in a principal open of some affine scheme U is principal open in U . Thus after replacing U by a suitable principal open, we may assume $U \subseteq V$. Now let $s \in \Gamma(V, \mathcal{O}_X)$ such that $D_V(s) \subseteq U \cap V$. This has the desired property since $D_V(s) = D_U(s|_U)$. (By assumption, $D_V(s) \subseteq U$, and we can check whether a point of U lies in this set by checking whether the image of s in the residue class field is $\neq 0$, and the residue class field does not depend on whether we compute it inside U or V .) $\square$

Lemma 1.20. *Let $A \rightarrow B$ be a ring homomorphism, and let $b_1, \dots, b_r \in B$ be a family of elements which generates the unit ideal and such that for every i the homomorphism $A \rightarrow B_{b_i}$ is of finite type. Then $A \rightarrow B$ is of finite type.*

Proof. For each i let $\left(\frac{c_{ij}}{b_i^{v_j}}\right)_j$ be a finite system of generators of B_{b_i} as an A -algebra, where $c_{ij} \in B$. Write $1 = \sum a_i b_i$. Let $B' \subseteq B$ be the A -subalgebra generated by all b_i , all a_i and all c_{ij} . We will show that $B' = B$, thus proving the lemma. So let $b \in B$. Its image in each B_{b_i} can be written as a polynomial expression in the $\frac{c_{ij}}{b_i^{v_j}}$ with coefficients in A , and clearing denominators, we find $N \geq 0$ such that $b_i^N b \in B'$. We may assume that the same N works for every i . Since B' contains the elements a_i , the b_i generate the unit ideal in B' , so the same is true for the b_i^N . Therefore the above implies that $b \in B'$, as desired. $\square$

Each of the properties of being *locally of finite type*, *quasi-compact*, and of *finite type* is stable under composition and under base change.

(1.6) Proper morphisms.

References: [GW1] Section (12.13); [H] II.4; [Mu] II.7.

Definition 1.21. Let $f: X \rightarrow Y$ be a morphism of schemes.

- (1) The morphism f is called *closed*, if for every closed subset $Z \subseteq X$, the image $f(Z)$ is a closed subset of Y .
- (2) The morphism f is called *universally closed*, if for every morphism $Y' \rightarrow Y$ the base change $X \times_Y Y' \rightarrow Y'$ of f along Y' is a closed morphism.
- (3) The morphism f is called *proper*, if it is separated, of finite type, and universally closed.

Example 1.22.

- (1) The affine line is not proper. More precisely, let k be a field, let $Y = \text{Spec}(k)$, and let $X = \mathbb{A}_k^1$. Let $f: X \rightarrow Y$ be the natural morphism. Then f is separated, of finite type and closed, but (why?) not universally closed.
- (2) Every closed immersion is proper.
- (3) The property of being proper is stable under composition and under base change.

Lemma 1.23.

- (1) Let $\mathbf{P}$ be a property of morphisms such that $\mathbf{P}$ is stable under composition and under base change and such that every closed immersion satisfies $\mathbf{P}$. If $g: X \rightarrow Y$ and $f: Y \rightarrow S$ are morphisms such that $f \circ g$ has $\mathbf{P}$ and such that f is separated, then g also satisfies $\mathbf{P}$.
- (2) Let S be a scheme, X a proper S -scheme and Y a separated S -scheme. Then every morphism $X \rightarrow Y$ of S -schemes is proper.

Proof. For (1), see Problem 2 on the problem sheets. Part (2) is the special case of (1), where $\mathbf{P}$ is the property of being proper. $\square$

(1.7) Projective schemes are proper.April 28,
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Definition 1.24. Let S be a scheme. An S -scheme X is called *projective* (we also say that X is *projective over S* , or that the morphism $X \rightarrow S$ is *projective*), if there exist $N \geq 0$ and a closed immersion $X \hookrightarrow \mathbb{P}_S^N$ of S -schemes.

This definition of *projective schemes* differs slightly from the one in [GW1] (Definition 13.68, which requires only that the above property holds locally on S). If S is affine, they coincide, however, and the difference will not be of any concern for us in this course. See [GW1], Summary 13.71 for a discussion. The definition given here is the one used in [H] and in [Stacks].

Suppose that $S = \operatorname{Spec} R$ is an affine scheme. For any homogeneous ideal $I \subseteq R[X_0, \dots, X_N]$, $V_+(I)$ is a closed subscheme of $\mathbb{P}_S^N$, and hence in particular a projective S -scheme. The following proposition shows that for S affine every projective scheme is isomorphic to a scheme of this form.

Proposition 1.25. (Problem 4 / [GW1] Proposition 13.24) Let R be a ring, $n \geq 0$, and let $Z \subseteq \mathbb{P}_R^n$ be a closed subscheme. Then there exists a homogeneous ideal $I \subseteq R[X_0, \dots, X_n]$ such that $Z \cong V_+(I)$.

Example 1.26. Let S be a scheme. Then every finite product of projective S -schemes is projective. To show this, since closed immersions are projective, it is enough to show that every finite product $\mathbb{P}_S^{n_1} \times_S \cdots \times_S \mathbb{P}_S^{n_r}$ of projective spaces over S is projective, and by induction (and applying the previous reduction again) it is sufficient to consider the case $r = 2$. Then one can use the *Segre embedding* (Problem 5).

For proving the main theorem of this section, we need the following lemma.

Lemma 1.27. Let R be a ring, $n \geq 0$, and let $I \subseteq (X_0, \dots, X_n) \subseteq R[X_0, \dots, X_n]$ be a homogeneous ideal. Then $V_+(I) = \emptyset$ if and only if $\operatorname{rad}(I) \supseteq (X_0, \dots, X_n)$.

Proof. It is easy to see that $V_+(I) = \emptyset$, if $(X_0, \dots, X_n) \subseteq \operatorname{rad}(I)$, either by showing that $V_+(I)$ then has no K -valued points for any field K , or by looking at the standard charts.

For the converse, fix $i \in \{0, \dots, n\}$. Now $V_+(I) \cap D_+(X_i) = \emptyset$ implies that the ideal (in $R \left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i} \right]$) generated by all $f/X_i^{\deg(f)}$, $f \in I$ homogeneous, contains 1. Writing this out and multiplying by a sufficiently high power of X_i , we see that $X_i \in \operatorname{rad}(I)$. $\square$

In addition, we will use the following commutative algebra lemma which is easily proved using the definitions of the localizations appearing in the lemma. (We will later generalize the lemma when we prove that given an $\mathcal{O}_X$ -module $\mathcal{F}$ of finite type on a locally ringed space X , the support of $\mathcal{F}$, i.e., the set of all points x such that the stalk $\mathcal{F}_x$ does not vanish, is closed. See Proposition 2.20.)

Lemma 1.28. *Let R be a ring, let $\mathfrak{p} \subset R$ be a prime ideal and let M be a finitely generated R -module. If the localization $M_{\mathfrak{p}}$ vanishes, then there exists $s \in R \setminus \mathfrak{p}$ such that already the localization M_s is zero.*

Theorem 1.29. ([GW1] Theorem 13.40) *Every projective morphism of schemes is proper.*

Sketch of proof. Since closed immersions are proper, it is enough to prove that projective space is proper, i.e., that for every scheme S the morphism $\mathbb{P}_S^n \rightarrow S$ is closed. Since this property can be checked locally on S , we may assume that $S = \operatorname{Spec} R$ is affine.

If $Z \subseteq \mathbb{P}_S^n$ is a closed subset, there exists a closed subscheme with underlying topological space Z , and hence (Proposition 1.25) a homogeneous ideal $I \subseteq R[X_0, \dots, X_n]$ such that $V_+(I)$ has underlying topological space Z . We need to show that the image of $V_+(I)$ in S is closed, or equivalently, that its complement $U \subseteq S$ is open.

Denote by f the composition $V_+(I) \hookrightarrow \mathbb{P}_S^n \rightarrow S$, and let $x \in U$. Then the scheme-theoretic fiber $f^{-1}(x) = V_+(I) \times_U \operatorname{Spec} \kappa(x)$ is empty. We want to show that there exists $s \in R$ such that $x \in D(s) \subseteq U$. The inclusion $D(s) \subseteq U$ amounts to saying that $f^{-1}(D(s)) = \emptyset$. To translate the problem into a commutative algebra statement, let $\bar{I}$ be the image of I in $\kappa(x)[X_0, \dots, X_n]$. It follows from the assumption $f^{-1}(x) = \emptyset$ and Lemma 1.27 that $\operatorname{rad}(\bar{I})$ contains the ideal $(X_0, \dots, X_n) (\subset \kappa(x)[X_0, \dots, X_n])$. Thus for d sufficiently large, for the degree d components we have $\bar{I}_d = (X_0, \dots, X_n)_d$. By the lemma of Nakayama, we obtain $I_d \otimes \mathcal{O}_{S,x} = (X_0, \dots, X_n)_d \subseteq \mathcal{O}_{S,x}[X_0, \dots, X_n]$. It then follows that the analogous equality holds already over the localization of R with respect to a suitable element s not contained in the prime ideal x . $\square$

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Remark 1.30. It may be illustrative to study what goes wrong when applying the above argument to “prove” that the morphism $\mathbb{A}_k^2 \rightarrow \mathbb{A}_k^1$ (for a field k) is closed (which we know is false, consider for instance the image of the hyperbola $V(ST - 1)$). The point is that for an ideal $I \subseteq K[S, T]$ and a point $t \in \mathbb{A}_k^1$, if the fiber over t is empty, i.e., the image of I in $\kappa(t)[S, T]$ generates the unit ideal, we cannot conclude that the same holds over $\mathcal{O}_{\mathbb{A}_k^1, t}$. The Lemma of Nakayama does not apply since the modules we need to compare ($I \otimes \mathcal{O}_{\mathbb{A}_k^1, t} \subseteq \mathcal{O}_{\mathbb{A}_k^1, t}[S]$) are not finitely generated over the local ring $\mathcal{O}_{\mathbb{A}_k^1, t}$.

Example 1.31. (The resultant of polynomials) Let k be an algebraically closed field (with some “obvious” adaptations, the results below hold over an arbitrary field). Let $m, n \in \mathbb{N}$. We identify the set A of pairs (f, g) of monic polynomials with the set of k -valued points of the affine space $\mathbb{A}_k^{m+n} = \operatorname{Spec} k[S_0, \dots, S_{m-1}, T_0, \dots, T_{n-1}]$, where a tuple $(s_i, t_j) \in k^{m+n} = \mathbb{A}_k^{m+n}(k)$ corresponds to $(X^m + s_{m-1}X^{m-1} + \dots + s_0, X^n + t_{n-1}X^{n-1} + \dots + t_0)$.

Viewing A as the set of closed points of $\mathbb{A}_k^{m+n}$, A is equipped with a topology, namely the topology induced by the Zariski topology.

Let $Z \subset A$ be the subset consisting of those pairs (f, g) such that f and g have a common zero in k . Write $R = k[S_0, \dots, S_{m-1}, T_0, \dots, T_{n-1}]$.

Claim. The set Z is a closed subset.

Proof of claim. Let

$$\begin{aligned} F &= X^m + S_{m-1}X^{m-1} + \dots + S_1X + S_0, \\ G &= X^n + T_{n-1}X^{n-1} + \dots + T_1X + T_0 \in R[X] \end{aligned}$$

be the “universal” monic polynomials, and let

$$\begin{aligned} \tilde{F} &= X^m + S_{m-1}X^{m-1}Y + \dots + S_1XY^{m-1} + S_0Y^m, \\ \tilde{G} &= X^n + T_{n-1}X^{n-1}Y + \dots + T_1XY^{n-1} + T_0Y^n \in R[X, Y] \end{aligned}$$

be their homogenizations with respect to a second variable Y .

Let $p: \mathbb{P}_R^1 \rightarrow \operatorname{Spec} R$ be the projection. Then $Z = p(V_+(\tilde{F}, \tilde{G})) \cap A$. By the above theorem, $p(V_+(\tilde{F}, \tilde{G}))$ is closed in $\mathbb{A}_k^{m+n}$, hence the claim follows. To see the equality, fix $x = (f, g) \in A$ and let $\tilde{f}, \tilde{g}$ be their homogenizations. Then f and g have a common zero in k if and only if $V_+(\tilde{f}, \tilde{g}) \neq \emptyset$ (inside $\mathbb{P}_{\kappa(x)}^1$). Note that the point $(1 : 0)$, the “point at infinity” in $\mathbb{P}^1$ is never a zero of $\tilde{f}$ or $\tilde{g}$. Since $V_+(\tilde{f}, \tilde{g}) = V_+(\tilde{F}, \tilde{G}) \times_{\operatorname{Spec} R} \operatorname{Spec} \kappa(x)$ can be identified with the (scheme-theoretic) fiber of p over the point x , this proves the desired description of Z .

More precisely one can show (using other methods) that Z is the zero locus of a single polynomial in R , the so-called *resultant* of a pair of monic polynomials. See [GW1] Section (B.20) for a sketch and further references, or [Bo] Abschnitt 4.4 for a detailed account in German.

2. $\mathcal{O}_X$ -MODULES

General references: [GW1] Ch. 7, [H] II.5.

Definition and basic properties.**(2.1) Definition of $\mathcal{O}_X$ -modules.**

Definition 2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$ -module is a sheaf $\mathcal{F}$ of abelian groups on X together with maps

$$\mathcal{O}_X(U) \times \mathcal{F}(U) \rightarrow \mathcal{F}(U) \quad \text{for each open } U \subseteq X$$

giving each $\mathcal{F}(U)$ the structure of an $\mathcal{O}_X(U)$ -module, and which are compatible with the restriction maps for open subsets $U' \subseteq U \subseteq X$.

An $\mathcal{O}_X$ -module homomorphism $\mathcal{F} \rightarrow \mathcal{G}$ between $\mathcal{O}_X$ -modules $\mathcal{F}, \mathcal{G}$ on X is a sheaf morphism $\mathcal{F} \rightarrow \mathcal{G}$ such that for all open subsets $U \subseteq X$, the map $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$ -modules. We denote the set of $\mathcal{O}_X$ -module homomorphisms from $\mathcal{F}$ to $\mathcal{G}$ by $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$; this is an $\mathcal{O}_X(X)$ -module (and in particular an abelian group).

We obtain the category $(\mathcal{O}_X\text{-Mod})$ of $\mathcal{O}_X$ -modules.

Remark 2.2. If $\mathcal{F}$ is an $\mathcal{O}_X$ -module and $x \in X$, then the stalk $\mathcal{F}_x$ carries a natural $\mathcal{O}_{X,x}$ -module structure and we obtain a functor from the category of $\mathcal{O}_X$ -modules to the category of $\mathcal{O}_{X,x}$ -modules. Now assume that X is a locally ringed space. The $\kappa(x)$ -vector space $\mathcal{F}(x) := \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is called the *fiber of $\mathcal{F}$ over x* . So an $\mathcal{O}_X$ -module on a locally ringed space gives us a family of vector spaces over the residue class fields.

Constructions, examples 2.3. Let X be a ringed space.

- (1) The zero sheaf and the structure sheaf $\mathcal{O}_X$ are $\mathcal{O}_X$ -modules in a natural way.
- (2) There is an obvious notion of $\mathcal{O}_X$ -module presheaf. (In the definition of $\mathcal{O}_X$ -module, replace *sheaf* by *presheaf*.) Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$ -modules. The multiplication by scalars is a bilinear map $\mathcal{O}_X \times \mathcal{F} \rightarrow \mathcal{F}$ of presheaves of abelian groups with certain further properties. Since sheafification is a functor and is compatible with finite products, we obtain a natural $\mathcal{O}_X$ -module structure on the sheafification of $\mathcal{F}$. This allows us to carry over most constructions we know for modules over rings to the setting of $\mathcal{O}_X$ -modules: First define a presheaf version, then, if necessary, pass to the sheafification.
- (3) Given an $\mathcal{O}_X$ -module $\mathcal{F}$, an $\mathcal{O}_X$ -submodule of $\mathcal{F}$ is an $\mathcal{O}_X$ -module $\mathcal{G}$ such that for every $U \subseteq X$ open, $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -submodule. The quotient $\mathcal{F}/\mathcal{G}$ then is defined as the sheafification of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$. It is an $\mathcal{O}_X$ -module in a natural way by (2).

- (4) We can construct direct sums $\bigoplus$, direct products $\prod$ (each with arbitrary index sets), tensor products of $\mathcal{O}_X$ -modules and limits and colimits of $\mathcal{O}_X$ -modules, by doing the corresponding presheaf construction and then sheafifying (if necessary; for products, hence also for direct sums, and limits, the corresponding presheaf is already a sheaf).
- (5) Similarly we can construct kernels, cokernels and image sheaves of homomorphism of $\mathcal{O}_X$ -modules. These are compatible with passing to the stalks, and we obtain a notion of exact sequence of $\mathcal{O}_X$ -modules (see below).
- (6) Restriction to open subsets: For $\mathcal{F}$ an $\mathcal{O}_X$ -modules and $U \subseteq X$ open, the restriction $\mathcal{F}_{X|U}$ is naturally an $\mathcal{O}_{X|U}$ -module.
- (7) The Hom sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, defined by $U \mapsto \text{Hom}_{\mathcal{O}_U}(\mathcal{F}_U, \mathcal{G}_U)$ (this is a sheaf, by “gluing of morphisms of sheaves”). For an $\mathcal{O}_X$ -module $\mathcal{F}$ we define the dual $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.
- (8) Tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. This is compatible with passing to stalks, i.e., we have

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x,$$

because tensor product commutes with colimits.

Note that taking products and the $\mathcal{H}om$ sheaf is not compatible with passing to the stalks in general. But compare Proposition 2.19/Problem 7.)

Definition 2.4. Let X be a ringed space.

- (1) A homomorphism $\mathcal{F} \rightarrow \mathcal{G}$ of $\mathcal{O}_X$ -modules is called *injective*, if for every $x \in X$, the induced map $\mathcal{F}_x \rightarrow \mathcal{G}_x$ is injective.
- (2) A homomorphism $\mathcal{F} \rightarrow \mathcal{G}$ of $\mathcal{O}_X$ -modules is called *surjective*, if for every $x \in X$, the induced map $\mathcal{F}_x \rightarrow \mathcal{G}_x$ is bijective.
- (3) A sequence $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}''$ of homomorphisms of $\mathcal{O}_X$ -modules is *exact at $\mathcal{F}$* if for all $x \in X$ the sequence $\mathcal{F}'_x \rightarrow \mathcal{F}_x \rightarrow \mathcal{F}''_x$ is exact.

We have seen in Algebraic Geometry 1 that a homomorphism is an isomorphism if and only if it is injective and surjective; and also that a homomorphism $\mathcal{F} \rightarrow \mathcal{G}$ is injective if and only if for all open subsets $U \subseteq X$, the map $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is injective, but that the analogous equivalence for surjective does not hold (requiring that all $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ are surjective implies that $\mathcal{F} \rightarrow \mathcal{G}$ is surjective, but is much stronger).

Since taking kernels and images is compatible with passing to stalks (i.e., the stalks of the kernel are the kernels of the homomorphisms induced on stalks, etc.) a sequence $\mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}''$ is exact at $\mathcal{F}$ if and only if $\ker(g) = \text{im}(f)$.

The category of $\mathcal{O}_X$ -modules is an abelian category.

An important class of examples of $\mathcal{O}_X$ -modules are *locally free* $\mathcal{O}_X$ -modules.

Definition 2.5. Let $\mathcal{F}$ be an $\mathcal{O}_X$ -module on a ringed space X . Call $\mathcal{F}$

- (a) *free*, if it is isomorphic to $\mathcal{O}_X^{(I)} := \bigoplus_{i \in I} \mathcal{O}_X$ for some set I ,

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(b) locally free, if there exists an open covering $X = \bigcup_j U_j$ of X such that $\mathcal{F}|_{U_j}$ is a free $\mathcal{O}_{U_j}$ -module for each j .

For $X \neq \emptyset$ the rank of a free $\mathcal{O}_X$ -module is the cardinality of I as above (we usually regard it in $\mathbb{Z} \cup \{\infty\}$, without making a distinction between infinite cardinals). It is determined by $\mathcal{F}$. The rank of a locally free $\mathcal{O}_X$ -module is a function $X \rightarrow \mathbb{Z} \cup \{\infty\}$ which is locally constant on X (i.e., on each connected component of X , there is an integer giving the rank).

An invertible sheaf or line bundle on X is a locally free sheaf of rank 1.

For $\mathcal{L}$ invertible, there is a natural isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$ (whence the name). (In fact, there is a natural homomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \rightarrow \mathcal{O}_X$ (this holds for any $\mathcal{O}_X$ -module), and we can check locally on X that it is an isomorphism. But then we may assume that $\mathcal{L} = \mathcal{O}_X$, and the assertion is clear.) Hence $\otimes$ induces the structure of a commutative group on the set of isomorphism classes of invertible sheaves in X (with neutral element the isomorphism class of the structure sheaf, also called the *trivial* line bundle). The resulting group is called the Picard group of X and denoted by $\text{Pic}(X)$.

(2.2) Inverse image.

Definition 2.6. Let $f: X \rightarrow Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be an $\mathcal{O}_X$ -module. Then $f_*\mathcal{F}$ carries a natural $\mathcal{O}_Y$ -module structure and is called the direct image or push-forward of $\mathcal{F}$ under f .

Definition 2.7. Let $f: X \rightarrow Y$ be a morphism of ringed spaces, $\mathcal{F}$ an $\mathcal{O}_Y$ -module.

We define

$$f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

We obtain functors f_* , f^* between the categories of $\mathcal{O}_X$ -modules and $\mathcal{O}_Y$ -modules.

For $x \in X$, we have $(f^*\mathcal{F})_x \cong \mathcal{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$. Since f^{-1} is exact, this shows that f^* is right exact. Recall also that f_* is left exact (on abelian sheaves, hence on $\mathcal{O}_X$ -modules). It is also immediate to check that for morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of ringed spaces, there is an equality $(g \circ f)_* = g_* \circ f_*$ of functors. With a bit of work one can check that likewise, there is an isomorphism $(g \circ f)^* \cong f^* \circ g^*$ of functors. This also follows formally from the adjunction between f^* and f_* .

Proposition 2.8. Let $f: X \rightarrow Y$ be a morphism of ringed spaces. The functors f_* is right adjoint to the functor f^* :

$$\text{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

for all $\mathcal{O}_X$ -modules $\mathcal{F}$, all $\mathcal{O}_Y$ -modules $\mathcal{G}$, functorially in $\mathcal{F}$ and $\mathcal{G}$.

Sketch of proof. We know already that for homomorphisms of abelian sheaves we have $\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$. This isomorphism restricts to an isomorphism $\mathrm{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$. On the other hand, similarly as for modules of rings, there are isomorphisms

$$\mathrm{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X, \mathcal{F}),$$

functorial in $\mathcal{F}$ and $\mathcal{G}$. $\square$

Quasi-coherent $\mathcal{O}_X$ -modules.

(2.3) The $\mathcal{O}_{\mathrm{Spec} A}$ -module attached to an A -module M .

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Definition 2.9. Let A be a ring and M an A -module. Then setting

$$D(f) \mapsto M_f, \quad f \in A,$$

is well-defined and defines a sheaf on the basis of principal open sets in $\mathrm{Spec} A$. We denote the corresponding sheaf on $\mathrm{Spec} A$ by $\widetilde{M}$. It is an $\mathcal{O}_{\mathrm{Spec} A}$ -module (by viewing each M_f as an A_f -module in the natural way).

Proving that the presheaf on the basis of principal opens is in fact a sheaf works in exactly the same way as for the structure sheaf of $\mathrm{Spec}(A)$. Similarly, the same proof as for the structure sheaf shows that the stalk of $\widetilde{M}$ at a point $\mathfrak{p} \in \mathrm{Spec} A$ is the localization $M_{\mathfrak{p}} = \left\{ \frac{m}{s}; m \in M, s \in A \setminus \mathfrak{p} \right\}$ (where fractions $\frac{m}{s}$ are to be understood as equivalence classes in the usual way). We may also identify $M_{\mathfrak{p}} = M \otimes_A A_{\mathfrak{p}}$.

Taking $M = A$, we obtain $\widetilde{A} = \mathcal{O}_{\mathrm{Spec}(A)}$.

Remark 2.10. For an affine scheme X , in general not every $\mathcal{O}_X$ -module has the above form. We will investigate this more closely soon.

Proposition 2.11. Let A be a ring, and let M, N be A -modules. Then the maps

$$\mathrm{Hom}_A(M, N) \rightarrow \mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec} A}}(\widetilde{M}, \widetilde{N})$$

given by

$$\varphi \mapsto \widetilde{\varphi} := (\varphi_f: M_f \rightarrow N_f)_f$$

and, in the other direction,

$$\Phi \mapsto \Gamma(\mathrm{Spec} A, \Phi),$$

are inverse to each other. In other words, $\widetilde{\cdot}$ is a fully faithful functor from the category of A -modules to the category of $\mathcal{O}_{\mathrm{Spec} A}$ -modules.

Proof. It is clear that $\Gamma(\mathrm{Spec}(A), \widetilde{\varphi}) = \varphi$. On the other hand, starting with Φ and defining $\varphi := \Gamma(\mathrm{Spec}(A), \Phi)$, for any $f \in A$ we consider the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\varphi} & N \\
\downarrow & & \downarrow \\
M_f & \longrightarrow & N_f.
\end{array}$$

This is commutative with lower horizontal homomorphism φ_f by functoriality of localization, but also with lower horizontal homomorphism $\Phi(D(f))$, since Φ is a morphism of sheaves. But the upper horizontal homomorphism determines the lower one uniquely, so it follows that $\Phi(D(f)) = \varphi_f$, as desired. $\square$

By applying the proposition to $M = A$, we also see that for an A -module N , $\widetilde{N}$ is zero if and only if N is zero.

The construction $M \mapsto \widetilde{M}$ is compatible with exactness, kernels, cokernels, images, direct sums, filtered inductive limits. More precisely:

Proposition 2.12. *Let A be a ring, $X = \operatorname{Spec}(A)$.*

- (1) *Let $M' \rightarrow M \rightarrow M''$ be a sequence of A -modules. The sequence is exact at M if and only if the sequence $\widetilde{M}' \rightarrow \widetilde{M} \rightarrow \widetilde{M}''$ is exact at $\widetilde{M}$.*
- (2) *Let $f: M \rightarrow N$ be a homomorphism of A -modules, and let $\widetilde{f}$ be the associated homomorphism of $\mathcal{O}_X$ -modules. Then $\ker(f)^\sim \cong \ker(\widetilde{f})$, $\operatorname{im}(f)^\sim \cong \operatorname{im}(\widetilde{f})$, $\operatorname{coker}(f)^\sim \cong \operatorname{coker}(\widetilde{f})$. In particular, f is injective (or surjective, respectively) if and only if $\widetilde{f}$ is injective (surjective).*
- (3) *The functor $\widetilde{}$ commutes with taking direct sums and colimits.*

Proof. For Part (1) use that the exactness of $M' \rightarrow M \rightarrow M''$ is equivalent to the exactness of $M'_\mathfrak{p} \rightarrow M_\mathfrak{p} \rightarrow M''_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$ (in fact, localization is an exact functor; and to go back one reduces to the statement that $M = 0$ if and only if all the localizations $M_\mathfrak{p} = 0$, $\mathfrak{p} \in \operatorname{Spec}(A)$).

Part (2) follows from (1). Part (3) follows from the compatibility of direct sums and colimits with localizations. $\square$

(2.4) Quasi-coherent modules.

Recall our notation $\mathcal{F}^{(I)} := \bigoplus_{i \in I} \mathcal{F}$.

Definition 2.13. *Let X be a ringed space. An $\mathcal{O}_X$ -module $\mathcal{F}$ is called quasi-coherent, if every $x \in X$ has an open neighborhood U such that there exists an exact sequence*

$$\mathcal{O}_U^{(J)} \rightarrow \mathcal{O}_U^{(I)} \rightarrow \mathcal{F}|_U \rightarrow 0$$

for suitable (possibly infinite) index sets I, J .

For a morphism $f: X \rightarrow Y$ of ringed spaces and a quasi-coherent $\mathcal{O}_Y$ -module $\mathcal{G}$, the pull-back $f^*\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$ -module (since f^{-1} is exact and tensor product is a right exact functor). The direct image f_* preserves the property of quasi-coherence (only) under certain conditions.

Locally free $\mathcal{O}_X$ -modules are quasi-coherent.

Clearly, for a ring A and an A -module M , $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_{\text{Spec } A}$ -module. We will see below that the converse is true as well:

For a ringed space X and $f \in \Gamma(X, \mathcal{O}_X)$, we write $X_f := \{x \in X; f_x \in \mathcal{O}_{X,x}^\times\}$, an open subset of X . We obtain a homomorphism

$$\Gamma(X, \mathcal{F})_f \rightarrow \Gamma(X_f, \mathcal{F})$$

for every $\mathcal{O}_X$ -module $\mathcal{F}$.

Theorem 2.14. *Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$ -module. The following are equivalent:*

- (i) *For every affine open $\text{Spec } A = U \subseteq X$, there exists an A -module M such that $\mathcal{F}|_U \cong \widetilde{M}$.*
- (ii) *There exists a covering $X = \bigcup_i U_i$ by affine open subschemes $U_i = \text{Spec } A_i$ and A_i -modules M_i such that $\mathcal{F}|_{U_i} \cong \widetilde{M}_i$ for all i .*
- (iii) *The $\mathcal{O}_X$ -module $\mathcal{F}$ is quasi-coherent.*
- (iv) *For every affine open $\text{Spec } A = U \subseteq X$ and every $f \in A$, the homomorphism $\Gamma(U, \mathcal{F})_f \rightarrow \Gamma(D(f), \mathcal{F})$ is an isomorphism.*

Note that we can phrase (iv) equivalently as saying that the natural map $\Gamma(U, \mathcal{F})^\sim \rightarrow \mathcal{F}|_U$ is an isomorphism.

Sketch of proof. The implications (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are relatively easy. To show (iii) $\Rightarrow$ (iv), we may assume $X = U = \text{Spec } A$ and we can cover X by finitely many principal open subsets $D(g_i)$ such that $\mathcal{F}|_{D(g_i)}$ is the cokernel of a homomorphism of free modules and hence (Proposition 2.12, Proposition 2.11) is of the form $\widetilde{M}_i$. In particular, (iv) holds for $\mathcal{F}|_{D(g_i)}$, and similarly for $\mathcal{F}|_{D(g_i g_j)}$. Now use the sheaf property of $\mathcal{F}$, and that localization is exact and commutes with finite products, to conclude that (iv) holds for U itself. $\square$

Corollary 2.15. *Let A be a ring, $X = \text{Spec } A$. The functor $\sim$ induces an exact equivalence between the categories of A -modules and of quasi-coherent $\mathcal{O}_X$ -modules.*

The statements of the following corollary can be checked locally on X , hence it is enough to show the corresponding claims for modules in the image of the $\sim$ functor. For Part (3) use that tensor product is compatible with localization.

Corollary 2.16. *Let X be a scheme.*

- (1) *Kernels, cokernels, images of $\mathcal{O}_X$ -module homomorphisms between quasi-coherent $\mathcal{O}_X$ -modules are quasi-coherent.*
- (2) *Direct sums of quasi-coherent $\mathcal{O}_X$ -modules are quasi-coherent.*
- (3) *Let $\mathcal{F}, \mathcal{G}$ be quasi-coherent $\mathcal{O}_X$ -module. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is quasi-coherent, and for every affine open $U \subseteq X$ we have*

$$\Gamma(U, \mathcal{F} \otimes \mathcal{G}) = \Gamma(U, \mathcal{F}) \otimes \Gamma(U, \mathcal{G}).$$

In particular, by (1) and (2) the category of quasi-coherent $\mathcal{O}_X$ -module is an abelian category, and the inclusion functor into the category of all $\mathcal{O}_X$ -modules preserves kernels and cokernels and direct sums.

(2.5) Direct and inverse image of quasi-coherent $\mathcal{O}_X$ -module.

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Proposition 2.17. *Let $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ be affine schemes, and let $f: X \rightarrow Y$ be a scheme morphism.*

- (1) *Let N be an B -module, then $f_*(\widetilde{N}) = \widetilde{N_{[A]}}$ where $N_{[A]}$ is N , considered as an A -module via $\Gamma(f): A \rightarrow B$.*
- (2) *Let M be an A -module, then $f^*(\widetilde{M}) = \widetilde{M \otimes_A B}$.*

Sketch of proof. The first part can easily be checked directly. For the second part, use that we already know that $f^*\widetilde{M}$ is quasi-coherent, the Yoneda lemma and adjunction (or in other words, uniqueness of the left adjoint functor of f_*). $\square$

(2.6) Finiteness conditions.

Definition 2.18. *Let X be a ringed space. We say that an $\mathcal{O}_X$ -module $\mathcal{F}$ is of finite type (or of finite presentation, resp.), if every $x \in X$ has an open neighborhood $U \subseteq X$ such that there exists $n \geq 0$ (or $m, n \geq 0$, resp.) and a short exact sequence*

$$\mathcal{O}_X^n \rightarrow \mathcal{F} \rightarrow 0$$

(or

$$\mathcal{O}_X^m \rightarrow \mathcal{O}_X^n \rightarrow \mathcal{F} \rightarrow 0,$$

resp.).

On an affine scheme, this coincides with the corresponding definitions in terms of modules (via $M \mapsto \widetilde{M}$). Note that every $\mathcal{O}_X$ -module of finite presentation is quasi-coherent.

Proposition 2.19. *Let X be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$ -module of finite presentation.*

- (1) *For all $x \in X$ and for each $\mathcal{O}_X$ -module $\mathcal{G}$, the canonical homomorphism of $\mathcal{O}_{X,x}$ -modules*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \rightarrow \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

is bijective.

- (2) *Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$ -modules of finite presentation. Let $x \in X$ be a point and let $\theta: \mathcal{F}_x \xrightarrow{\sim} \mathcal{G}_x$ be an isomorphism of $\mathcal{O}_{X,x}$ -modules. Then there exists an open neighborhood U of x and an isomorphism $u: \mathcal{F}|_U \xrightarrow{\sim} \mathcal{G}|_U$ of $\mathcal{O}_U$ -modules with $u_x = \theta$.*

Proof. Problem 7. □

Proposition 2.20. *Let X be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$ -module of finite type. Then the support*

$$\text{Supp}(\mathcal{F}) = \{x \in X; \mathcal{F}_x \neq 0\}$$

of $\mathcal{F}$ is a closed subset of X .

(2.7) Closed subschemes and quasi-coherent ideal sheaves.

Proposition 2.21. *Let X be a scheme. An ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ defines a closed subscheme if and only if $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$ -module.*

The point here is that both properties can be checked locally on X . For affine schemes we have already shown this statement in Algebraic Geometry 1 (which amounts to saying that every closed subscheme of an affine scheme $\text{Spec } A$ has the form $\text{Spec } A/\mathfrak{a}$ for some ideal $\mathfrak{a} \subseteq A$). However, with the new tools we can give a direct proof, and as a corollary again get the statement for affine schemes.

Proof. Let $\mathcal{I}$ be a quasi-coherent ideal sheaf. Then $Z := \text{Supp}(\mathcal{O}_X/\mathcal{I})$ is a closed subset of X and we need to check that the locally ringed space $(Z, i^{-1}(\mathcal{O}_X/\mathcal{I}))$ is a scheme (where $i: Z \rightarrow X$ denotes the inclusion map). This can be done locally on X , so we may assume that $X = \text{Spec}(A)$ is affine, and hence that $\mathcal{I}$ is of the form $\tilde{\mathfrak{a}}$ for some ideal $\mathfrak{a} \subseteq A$. But then we may identify $(Z, i^{-1}(\mathcal{O}_X/\mathcal{I}))$ with $\text{Spec}(A/\mathfrak{a})$ which obviously is a scheme.

Conversely, let $i: Z \rightarrow X$ be a closed immersion with corresponding ideal sheaf $\mathcal{I} = \text{Ker}(\mathcal{O}_X \rightarrow i_*\mathcal{O}_Z)$, and let us show that $\mathcal{I}$ is quasi-coherent. Again we may work locally on X , i.e., we may assume that $X = \text{Spec}(A)$ is an affine scheme. Moreover, we may also assume that Z is affine. In fact, since Z is a scheme, we find an open cover $X = \bigcup U_i$ of X such that all $Z \cap U_i$ are affine. At this point we do not know whether the U_i are affine but if $D(f) \subseteq U_i \subseteq X$ is a principal open in X , then $Z \cap D(f)$ is a principal open in $Z \cap U_i$, so also affine. Thus passing to a refinement of the original cover, if necessary, we find an *affine* open cover $X = \bigcup U_i$ such that all $Z \cap U_i$ are affine.

So now consider the situation that X and Z are both affine. But then $i: Z \rightarrow X$ is a morphism of affine schemes, and since $\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_Z$ -module, the direct image $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$ -module by Proposition 2.17. So $\mathcal{I} = \text{ker}(\mathcal{O}_X \rightarrow i_*\mathcal{O}_Z)$ is quasi-coherent, as well. □

Corollary 2.22. *Let $X = \text{Spec}(A)$ be an affine scheme and $Z \subseteq X$ a closed subscheme. Then $Z \cong \text{Spec}(A/\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq A$.*

We hence obtain an inclusion-reversing bijection between the set of closed subschemes of a scheme X and the set of quasi-coherent ideal sheaves in $\mathcal{O}_X$, mapping

- a quasi-coherent ideal sheaf $\mathcal{I}$ to $Z := (\text{Supp}(\mathcal{O}_X/\mathcal{I}), i^{-1}(\mathcal{O}_X/\mathcal{I}))$, where $i: \text{Supp}(\mathcal{O}_X/\mathcal{I}) \rightarrow X$ denotes the inclusion,
- a closed subscheme $Z \subseteq X$ to $\text{Ker}(\mathcal{O}_X \rightarrow i_*\mathcal{O}_Z)$, where $i: Z \rightarrow X$ denotes the inclusion morphism.

We denote the closed subscheme corresponding to a quasi-coherent ideal sheaf $\mathcal{I}$ by $V(\mathcal{I})$.

(2.8) Locally free sheaves on affine schemes.

There is an obvious “commutative algebra way” of writing down, for an A -module M , the condition that $\widetilde{M}$ is locally free.

Theorem 2.23. *Let A be a ring and M an A -module. Consider the following properties of M :*

- (i) $\widetilde{M}$ is a locally free $\mathcal{O}_{\text{Spec } A}$ -module.
 - (ii) M is locally free, i.e., there exist $f_1, \dots, f_n \in A$ generating the unit ideal such that for all i , the A_{f_i} -module M_{f_i} is free.
 - (iii) For all $\mathfrak{p} \in \text{Spec } A$, the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is free.
 - (iv) The A -module M is flat.
- (1) We have the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).
(2) If M is an A -module of finite presentation, then all the four properties are equivalent.

Proof. Part (1) is easy. Part (2) is more difficult. The implication (iii) $\Rightarrow$ (ii), for finitely presented M , follows from Prop. 2.19. See [GW1] Prop. 7.40.

There are several ways to show (iv) $\Rightarrow$ (iii). One can proceed in a fairly elementary fashion, using the “equational criterion of flatness”, see [M2] Theorem 7.10, Theorem 7.12. Alternatively, using the Tor functor (the “left derived functor of the tensor product”), one can proceed as follows. We may assume that A is local with maximal ideal $\mathfrak{m}$. Lift the elements of an $A/\mathfrak{m}$ -basis of $M/\mathfrak{m}M$ to M . By the Lemma of Nakayama this induces a surjection $A^n \rightarrow M$. Let K denote its kernel. We want to show that $K = 0$. Since M is of finite presentation, K is a finitely generated A -module ([M2] Theorem 2.6). Furthermore, the flatness of M implies that $\text{Tor}_1^A(A/\mathfrak{m}, M) = 0$. Therefore the short exact sequence

$$0 \longrightarrow K \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

remains exact after tensoring $- \otimes_A A/\mathfrak{m}$. Since by construction the homomorphism $A^n \rightarrow M$ becomes an isomorphism after tensoring with the residue class field, this shows that $K \otimes_A A/\mathfrak{m} = 0$. Applying the Lemma of Nakayama again, we obtain that $K = 0$, as desired. $\square$

There is an obvious analogous theorem for $\mathcal{O}_X$ -modules on a scheme X , where we define

Definition 2.24. Let X be a scheme. An $\mathcal{O}_X$ -module $\mathcal{F}$ is called *flat*, if for all $x \in X$ the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$ -module.

More generally, given an $\mathcal{O}_X$ -module $\mathcal{F}$ and a morphism $f: X \rightarrow Y$ we say that $\mathcal{F}$ is *f-flat* or *flat over Y*, if for all $x \in X$ the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$ -module (via $f_x^\sharp: \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$).

If A is a domain, then every flat A -module M is torsion-free (i.e., multiplication by s is injective for all $s \in A \setminus \{0\}$). The converse holds only rarely; it does hold if A is a principal ideal domain and M is finitely generated.

Remark 2.25.

- (1) Let A be a principal ideal domain. Then every finitely generated locally free (in the sense of condition (i') in the theorem) A -module is free. (Use the structure theorem for finitely generated modules over principal ideal domains.)
- (2) It is a difficult theorem (conjectured by Serre, proved independently by Quillen and Suslin) that every locally free sheaf of finite type on $\mathbb{A}_k^n$, k a field, is free. The same statement holds even for k a discrete valuation ring.
- (3) It will not be relevant in the course, but in fact in the previous two items the hypothesis *of finite type* can be omitted. In fact, whenever R is a ring which is noetherian and such that $\text{Spec } R$ is connected, then every locally free R -module which is *not finitely generated* is free. One way to show this is to combine the paper [Ba] by H. Bass with the difficult theorem that the property of a module of being “projective” can be checked Zariski-locally on $\text{Spec } A$ ([Stacks] 058B), which shows that all locally free R -modules, finitely generated or not, are projective. Maybe there is also a more direct way, without talking about projective modules?
- (4) On the other hand, even for an affine scheme X , a locally free $\mathcal{O}_X$ -module is usually not a projective object in the category $(\mathcal{O}_X\text{-Mod})$.
- (5) Let A be a noetherian unique factorization domain. Then every invertible sheaf on $\text{Spec } A$ is free.
- (6) See the answers to this question (mathoverflow.net/q/54356¹) for examples of non-free locally free modules over $\text{Spec } A$ for factorial (and even, in addition, regular) noetherian rings A .
- (7) Let A be a domain, and let M be a locally free A -module of rank 1. Then M is isomorphic to a *fractional ideal*, i.e., to a finitely generated sub- A -module of $K := \text{Frac}(A)$. (Cf. Problem 8 for a converse statement in the case that A is a Dedekind domain.)

¹<https://mathoverflow.net/q/54356>

3. LINE BUNDLES AND DIVISORS

General references: [GW1] Ch. 11, in particular (11.9), (11.13); [H] II.6.

A *divisor* on a scheme X should be thought of an object that encodes a “configuration of zeros and poles (with multiplicities)” that a function on X could have. Below, we will see two ways to make this precise and compare them.

Let X be an integral (i.e., reduced and irreducible) scheme. We denote by $K(X)$ the field of rational functions of X .

Later we will impose the additional condition that X is noetherian and that all local rings $\mathcal{O}_{X,x}$ are unique factorization domains.

An important example that is good to keep in mind is the case of a *Dedekind scheme of dimension 1*, i.e., X is a noetherian integral scheme such that all points except for the generic point are closed, and such that for every closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a principal ideal domain (in other words: all local rings are discrete valuation rings), and the generic point is not closed itself. If a Dedekind scheme X is a k -scheme of finite type for some algebraically closed (or at least perfect) field k , then we call X a smooth algebraic curve over k .

(3.1) Preliminaries.

References: [GW1] Sections (3.10), (3.11).

We start by defining some general notions that will become important in this chapter.

Recall the definition of a reduced ring:

Definition 3.1. *Let R be a ring. An element $x \in R$ is called nilpotent, if there exists $n \geq 0$ such that $x^n = 0$. The ring R is called reduced, if it has no nilpotent elements $\neq 0$.*

Definition/Proposition 3.2. *A scheme X is called reduced, if the following equivalent conditions are satisfied:*

- (i) *For every open $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is reduced.*
- (ii) *For every affine open $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is reduced.*
- (iii) *There exists an affine open cover $X = \bigcup_i U_i$ such that for every i the ring $\Gamma(U_i, \mathcal{O}_X)$ is reduced.*
- (iv) *For every $x \in X$, the ring $\mathcal{O}_{X,x}$ is reduced.*

Every domain is reduced, but not conversely. More precisely, a reduced ring R is a domain if and only if it has a *unique* minimal prime ideal (necessarily the zero ideal), if and only if $\text{Spec}(R)$ is irreducible. This leads to the notion of *integral* scheme, see below. Before we come to it, we discuss irreducibility and generic points in the context of schemes.

Definition 3.3. A scheme X is called *irreducible*, if its underlying topological space is irreducible (i.e., is $\neq \emptyset$ and cannot be written as a union of two proper closed subsets).

From the corresponding result for affine schemes, we obtain the existence and uniqueness of generic points.

Proposition 3.4. Let X be a scheme, and let $Z \subseteq X$ be an irreducible closed subset of (the underlying topological space of) X . Then Z has a unique generic point, i.e., there exists a unique $\eta \in Z$ such that $Z = \{\eta\}$.

Proof. We have shown in Algebraic Geometry 1 that this is true for affine schemes X . Now let X be any scheme, $Z \subseteq X$ an irreducible closed subset, and $U \subseteq X$ an affine open which intersects Z . Then $U \cap Z$ is dense in Z , since Z is irreducible. First this implies that $U \cap Z$ is irreducible. It is clearly closed in the affine scheme U , and therefore has a generic point η . Using again that $U \cap Z$ is dense in Z , we see that the closure of $\{\eta\}$ in Z is all of Z . Furthermore, all generic points of Z are contained in $U \cap Z$, and therefore coincide by the uniqueness statement in the affine case. $\square$

For non-irreducible spaces, the following notion is useful:

Definition 3.5. The maximal irreducible subsets of a topological space X are called the *irreducible components* of X .

Since the closure of an irreducible subset is again irreducible, irreducible components are closed. Every irreducible subset is contained in an irreducible component (use that the union of a chain of irreducible subsets is irreducible and Zorn's lemma), so every topological space is the union of its irreducible components. (Without further finiteness conditions, this notion is however not very well-behaved.)

Definition/Proposition 3.6. A scheme X is called *integral*, $X \neq \emptyset$ and if the following equivalent conditions are satisfied:

- (i) The scheme X is reduced and irreducible.
- (ii) For every non-empty open $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is a domain.
- (iii) For every non-empty affine open $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is a domain.

Proof. By the above discussion, an affine scheme $\text{Spec}(R)$ is reduced and irreducible if and only if R is a domain. If X is a reduced (or irreducible, respectively) scheme, then so is every non-empty open subscheme. Together we obtain (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) is trivial. Regarding (iii) $\Rightarrow$ (i), we have already seen that (iii) implies that X is reduced.

It remains to show that (iii) implies that X is irreducible. Otherwise, there exist non-empty open subsets $U, U' \subseteq X$ with empty intersection. By shrinking U and U' , if necessary, we may assume that U and U' are affine open subschemes. Then $U \cup U'$ is a non-irreducible affine open subscheme, so $\Gamma(U \cup U', \mathcal{O}_X)$ is not a domain. $\square$

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If X is an integral scheme, then all the local rings $\mathcal{O}_{X,x}$ are domains (note that the converse does not hold, though; cf. Example 3.9 (2)). Since X is irreducible, it has a (unique) generic point $\eta \in X$. The local ring $K(X) := \mathcal{O}_{X,\eta}$ is a domain with only one prime ideal, and hence a field, called the *field of rational functions of X* .

For example, if k is a field, then $K(\mathbb{A}_k^1) = k(T)$, the field of fractions of the polynomial ring $k[T]$. This explains the name *field of rational functions*.

This construction is functorial for dominant maps:

Definition 3.7. *A morphism $f: X \rightarrow Y$ of schemes is called dominant, if the image of the continuous map f is dense in Y . If X and Y are irreducible, it is equivalent to saying that the generic point of X is mapped to the generic point of Y .*

We obtain contravariant functor from the category of integral schemes with scheme morphisms as morphisms to the category of fields, by mapping

$$X \mapsto K(X)$$

and mapping a dominant morphism $f: X \rightarrow Y$ between integral schemes X , Y with generic points η_X , η_Y , to

$$K(Y) = \mathcal{O}_{Y,\eta_Y} = \mathcal{O}_{Y,f(\eta_X)} \rightarrow \mathcal{O}_{X,\eta_X} = K(X).$$

The homomorphism $\mathcal{O}_{Y,f(\eta_X)} \rightarrow \mathcal{O}_{X,\eta_X}$ is the homomorphism of local rings induced by $f^\flat: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$.

Definition 3.8.

- (1) *A topological space is called noetherian, if it satisfies the descending chain condition for closed subsets, i.e., every chain*

$$Z_0 \supseteq Z_1 \supseteq \cdots$$

of closed subsets of X stabilizes (there exists n_0 such that $Z_n = Z_{n_0}$ for all $n \geq n_0$).

- (2) *A scheme X is called locally noetherian, if there exists an affine open cover $X = \bigcup_i U_i$ such that for every i the ring $\Gamma(U_i, \mathcal{O}_X)$ is noetherian (i.e., satisfies the ascending chain condition for ideals; equivalently, every ideal is finitely generated).*
- (3) *A scheme is called noetherian, if it is locally noetherian and quasi-compact.*

Remark 3.9. See [GW1] Sections (1.7), (3.9) for details.

- (1) A topological space X is noetherian if and only if every non-empty set of closed subsets of X has a minimal element, if and only if every non-empty set of open subsets of X has a maximal element, if and only if every open subset is quasi-compact.
- (2) A noetherian topological space has only finitely many irreducible components. (Otherwise the set of closed subsets that cannot be written as a finite union of irreducible sets is non-empty and thus contains a minimal

element Z . But then Z is not irreducible, and writing it as a union of proper closed subsets leads to a contradiction to the minimality.) Using this, one may show that a *connected noetherian* scheme X such that for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a domain, is integral. (One cannot drop the hypothesis of X being noetherian here, see [Stacks] Example 0568.

- (3) Every subspace of a noetherian topological space is noetherian.
- (4) If X is a locally noetherian scheme, and $U \subseteq X$ affine open, then $\Gamma(U, \mathcal{O}_X)$ is a noetherian ring. In particular, an affine scheme $\text{Spec}(A)$ is noetherian if and only if A is a noetherian ring.
- (5) If X is a noetherian scheme, then the underlying topological space is noetherian. (The converse is not true.)

Definition 3.10.

- (1) Let X be a topological space. The (Krull) dimension of X is the supremum over all ℓ such that there exists a chain

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_\ell \subseteq X$$

with $Z_i \subseteq X$ closed irreducible. (For $X = \emptyset$ we define $\dim(\emptyset) = -\infty$.)

- (2) For a scheme X , we define the dimension $\dim(X)$ as the dimension of the underlying topological space.
- (3) For a ring A , $\dim(\text{Spec}(A))$ is called the (Krull) dimension of the ring A .

Example 3.11.

- (1) Easy: Every field has dimension 0. Every principal ideal domain which is not a field has dimension 1.
- (2) More difficult: If k is a field, then $\dim(k[T_1, \dots, T_n]) = n$. Using Noether normalization, one can then show that more generally, if A is a finitely generated k -algebra which is a domain, then $\dim(A) = \text{trdeg}(\text{Frac}(A)/k)$, the transcendence degree of $\text{Frac}(A)$ over k (i.e., the minimal d such that there exist $x_1, \dots, x_d \in \text{Frac}(A)$ such that $\text{Frac}(A)/k(x_1, \dots, x_d)$ is an *algebraic* extension). See [GW1] Section (5.6).

Cartier divisors.

(3.2) Cartier divisors: Definition.

Denote by $K(X) = \mathcal{O}_{X,\eta}$ the field of rational functions on the integral scheme X , where $\eta \in X$ is the generic point. We denote by $\mathcal{K}_X$ the constant sheaf with value $K(X)$, i.e., $\mathcal{K}_X(U) = K(X)$ for all $\emptyset \neq U \subseteq X$ open. Since X is irreducible, this is a sheaf.

The notion of Cartier divisor encodes a zero/pole configuration by specifying, *locally on X* , functions with the desired zeros and poles. Since functions

which are units in $\Gamma(U, \mathcal{O}_X)$ should be regarded as having no zeros and/or poles on U , we consider functions only up to units.

Definition 3.12. A Cartier divisor on X is given by a tuple $(U_i, f_i)_i$, where $X = \bigcup_i U_i$ is an open cover, $f_i \in K(X)^\times$, and $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X)^\times$ for all i, j . Two such tuples $(U_i, f_i)_i, (V_j, g_j)_j$ give rise to the same divisor, if $f_i g_j^{-1} \in \Gamma(U_i \cap V_j, \mathcal{O}_X)^\times$ for all i, j .

With addition given by

$$(U_i, f_i)_i + (V_j, g_j)_j = (U_i \cap V_j, f_i g_j)_{i,j}$$

the set $\text{Div}(X)$ of all Cartier divisors on X is an abelian group.

Remark 3.13. We have $\text{Div}(X) = \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$.

Definition 3.14. A Cartier divisor of the form (X, f) , $f \in K(X)^\times$, is called a principal divisor. Divisors D, D' on X are called linearly equivalent, if $D - D'$ is a principal divisor. The set of principal divisors is a subgroup of $\text{Div}(X)$ and the quotient $\text{DivCl}(X)$ of $\text{Div}(X)$ by this subgroup is called the divisor class group of X .

(3.3) The line bundle attached to a Cartier divisor.

Let D be a Cartier divisor on X . We define an invertible $\mathcal{O}_X$ -module $\mathcal{O}_X(D)$ as follows:

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in K(X); \forall i : f_i f \in \Gamma(U \cap U_i, \mathcal{O}_X)\} \quad \text{for } \emptyset \neq U \subseteq X \text{ open.}$$

For each i , we have $\mathcal{O}_X(D)|_{U_i} = f_i^{-1} \mathcal{O}_{U_i} \subset \mathcal{K}_X$, so multiplication by f_i gives an $\mathcal{O}_{U_i}$ -module isomorphism $\mathcal{O}_X(D)|_{U_i} \cong \mathcal{O}_{U_i}$.

Proposition 3.15. The map $D \mapsto \mathcal{O}_D(X)$ induces group isomorphisms $\text{Div}(X) \cong \{\mathcal{L} \subset \mathcal{K}_X \text{ invertible } \mathcal{O}_X\text{-module}\}$ and $\text{DivCl}(X) \cong \text{Pic}(X)$.

Sketch of proof. To construct an inverse of the map $D \mapsto \mathcal{O}_D(X)$, take $\mathcal{L} \subseteq \mathcal{K}_X$ invertible and choose an open cover $X = \bigcup U_i$ such that $\mathcal{L}|_{U_i}$ is trivial for each i . Then necessarily $\mathcal{L}|_{U_i} = f_i^{-1} \mathcal{O}_{U_i}$ for some $f_i \in K(X)^\times$ (namely, f_i^{-1} is the image of $1 \in \Gamma(U_i, \mathcal{O}_X)$ under the map $\Gamma(U_i, \mathcal{O}_X) \rightarrow K(X)$ induced by the composition $\mathcal{O}_{U_i} \cong \mathcal{L}|_{U_i} \rightarrow \mathcal{K}_X$). We then map $\mathcal{L}$ to the Cartier divisor $(U_i, f_i)_i$. One checks that this map is well-defined (i.e., independent of the choice of cover and of the choice of the elements f_i) and that the two maps are inverse to each other.

It remains to check that $\mathcal{O}_D(X)$ is free if and only if D is principal (this follows easily from the definitions) and that every invertible $\mathcal{O}_X$ -module $\mathcal{L}$ can be embedded as a submodule into $\mathcal{K}_X$. This is easy if $X = \text{Spec}(A)$ is affine: If M is a locally free A -module of rank 1, then $M \otimes_A \text{Frac}(A)$ is a $\text{Frac}(A)$ -vector space of dimension 1 and we obtain the embedding $M \rightarrow M \otimes_A \text{Frac}(A) \cong \text{Frac}(A)$. (The map $M = M \otimes_A A \rightarrow M \otimes_A \text{Frac}(A)$

is injective since M , being locally free, is a flat A -module.) In the general case, let $U \subseteq X$ be open affine. We claim that every embedding $\mathcal{L}|_U \hookrightarrow \mathcal{K}_U$ extends uniquely to an embedding $\mathcal{L} \hookrightarrow \mathcal{K}_X$. Because of the uniqueness, we can work locally on X (and afterwards use gluing of sheaf homomorphisms), and therefore restrict to the case $\mathcal{L} = \mathcal{O}_X$. The embedding $\mathcal{O}_U = \mathcal{L}|_U \hookrightarrow \mathcal{K}_U$ then corresponds to a section $s \in \Gamma(U, \mathcal{K}_U)^\times$. But $\Gamma(U, \mathcal{K}_U) = K(X) = \Gamma(X, \mathcal{K}_X)$, so the claim follows. (See [GW1] Prop. 11.29 for more details and a variant which does not require X to be integral.) $\square$

To get a more geometric view on divisors, a first step is the following definition of the support of a divisor. We will carry this further by introducing the notion of Weil divisor, see below, and relating it to Cartier divisors.

Definition 3.16. *The support of a Cartier divisor D is*

$$\text{Supp}(D) = \{x \in X; f_{i,x} \in K(X)^\times \setminus \mathcal{O}_{X,x}^\times \text{ (where } x \in U_i)\},$$

a proper closed subset of X .

Weil divisors.

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(3.4) Normal schemes.

We will now study a different, more geometric, perspective on divisors, namely the notion of *Weil divisor*. In order to obtain a better behaved theory we make the assumption that the scheme X is noetherian, integral and *normal* in the sense of the following definition.

Definition 3.17. *A scheme X is called normal if all the local rings $\mathcal{O}_{X,x}$ are integrally closed domains.*

Recall that a domain A is called *integrally closed*, if every element of the field of fractions of A that is a zero of some monic polynomial with coefficients in A lies in A . Every unique factorization domain is normal. The property of being integrally closed is compatible with localizations, therefore we have (see [AM] Prop. 5.13 for more details):

Lemma 3.18. *Let A be a domain. Then $\text{Spec}(A)$ is normal if and only if A is integrally closed.*

For noetherian local domains of dimension 1, we have the following equivalence:

Proposition 3.19. *Let A be a noetherian local domain with $\dim(A) = 1$. The following are equivalent:*

- (i) *The ring A is normal.*
- (ii) *The ring A is a principal ideal domain.*

- (iii) *The ring A is a discrete valuation ring, i.e., there exists a valuation $v: \text{Frac}(A)^\times \rightarrow \mathbb{Z}$ such that $v(xy) = v(x) + v(y)$ and $v(x + y) \geq \min(v(x), v(y))$ for all $x, y \in \text{Frac}(A)$ and such that*

$$A = \{x \in K; v(x) \geq 0\} \cup \{0\}.$$

(Then A is local with maximal ideal $\{x \in K; v(x) > 0\} \cup \{0\}$.)

If v is a valuation, we set $v(0) = \infty$.

Sketch of proof. Every principal ideal domain is a UFD and hence normal by the remark above. If A is a principal ideal domain with maximal ideal $\mathfrak{m}$, one defines a valuation v on A by $v(x) = \max\{n; x \in \mathfrak{m}^n\}$ for $x \in A$ and extends this to K by the rule $v(xy) = v(x) + v(y)$. Conversely, A is the discrete valuation ring for the valuation v and $t \in A$ an element with $v(t) = 1$, one shows that the non-zero ideals of A are precisely the ideals (t^i) , $i \in \mathbb{N}$.

It remains to show that every normal noetherian local domain of dimension 1 is a discrete valuation ring. See, e.g., [AM] Proposition 9.2. $\square$

Passing to not necessarily local rings, we have the notion of *Dedekind domain* and *Dedekind scheme*:

Definition 3.20.

- (1) *A noetherian domain A is called a Dedekind domain if it is of dimension ≤ 1 and integrally closed (equivalently: all localizations $A_{\mathfrak{m}}$ at maximal ideals are principal ideal domains).*
- (2) *A noetherian integral scheme X is called a Dedekind scheme if for every affine open $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is a Dedekind domain.*

Also recall that a noetherian local ring A is called regular, if its maximal ideal can be generated by $\dim A$ elements, where $\dim A$ is the Krull dimension of A . If $\dim A = 0$, then being regular is equivalent to being a field. If $\dim A = 1$, then being regular is equivalent to being normal. In general, every regular local ring is normal, but not conversely.

One can show (Theorem of Auslander-Buchsbaum, [M2] Theorem 20.3) that every *regular local* noetherian ring is a UFD.

(3.5) Definition of Weil divisors.

Now let X be a normal noetherian integral scheme. (The theory can be set up in more generality, see [GW1] Section (11.13).)

Let $Z^1(X)$ denote the free abelian group on maximal proper integral subschemes of X (equivalently: those integral subschemes $Z \subset X$ such that for the generic point $\eta_Z \in Z$ we have $\dim \mathcal{O}_{X, \eta_Z} = 1$). We say that Z has *codimension 1*. We also write $\mathcal{O}_{X, Z} := \mathcal{O}_{X, \eta_Z}$.

By our assumptions on X , all the rings $\mathcal{O}_{X, Z}$ are discrete valuation rings. We denote by $v_Z: K(X)^\times \rightarrow \mathbb{Z}$ the corresponding discrete valuation on K , and set $v_Z(0) = \infty$.

Definition 3.21. An element of $Z^1(X)$ is called a Weil divisor. We write Weil divisors as finite “formal sums” $\sum n_Z[Z]$ where $Z \subset X$ runs through the integral closed subschemes of X of codimension 1.

For $f \in K(X)^\times$, we define the divisor attached to f as

$$\operatorname{div}(f) = \sum_Z v_Z(f)[Z].$$

Note that the sum is finite, i.e., $v_Z(f) = 0$ for all but finitely many Z . In fact, for $U \subseteq X$ affine open, the complement $X \setminus U$ has only finitely many irreducible components, so we may discard it and replace X by U . Then assume $X = \operatorname{Spec} A$ is affine and write $f = g/h$ with $g, h \in A$. Then $v_Z(f)$ can only be $\neq 0$, if Z is an irreducible component of $V(g) \cup V(h)$. Since this closed subscheme of the noetherian scheme X has only finitely many irreducible components (being itself noetherian), we are done.

Weil divisors of the form $\operatorname{div}(f)$ are called *principal Weil divisors*. The map div is a group homomorphism $K(X)^\times \rightarrow Z^1(X)$ and we call its cokernel, the quotient $\operatorname{Cl}(X) := Z^1(X)/\operatorname{im}(\operatorname{div})$, the (*Weil divisor*) *class group* of X . Two Weil divisors are called *linearly equivalent*, if their difference is a principal divisor; in other words, if they have the same image in $\operatorname{Cl}(X)$.

Definition 3.22.

- (1) A Weil divisor $\sum_Z n_Z[Z]$ is called *effective*, if $n_Z \geq 0$ for all Z .
- (2) A Cartier divisor D is called *effective*, if $\mathcal{O}_X \subseteq \mathcal{O}_X(D)$ (inside $\mathcal{K}_X$), or equivalently, if $\mathcal{O}_X(-D) \subseteq \mathcal{O}_X$ is an ideal of $\mathcal{O}_X$.

(3.6) Weil divisors vs. Cartier divisors.

Generalizing the definition of principal divisors, we can construct a group homomorphism $\operatorname{cyc}: \operatorname{Div}(X) \rightarrow Z^1(X)$ as follows:

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$$D = (U_i, f_i) \mapsto \sum v_Z(f_{i_Z})[Z],$$

where for each Z we choose an index i_Z so that U_{i_Z} contains the generic point of Z (equivalently: $U_{i_Z} \cap Z \neq \emptyset$).

The map cyc is injective, and is even an isomorphism $\operatorname{Div}(X) \cong Z^1(X)$, if X is locally factorial, i.e., all local rings $\mathcal{O}_{X,x}$ are UFD's. To prove this, we need the following facts from commutative algebra.

Lemma 3.23.

- (1) Let A be an integrally closed domain. Then A is equal to the intersection of all localizations $A_{\mathfrak{p}}$ (in $\operatorname{Frac}(A)$), where $\mathfrak{p}$ runs through the set of minimal non-zero prime ideals of A .
- (2) Let A be a unique factorization domain. Then every prime ideal $\mathfrak{p}$ of height 1 (i.e., $\dim A_{\mathfrak{p}} = 1$, equivalently $\mathfrak{p} \neq 0$ and $\mathfrak{p}$ is minimal among all prime ideals $\neq 0$ of A) is a principal ideal.

Proof. (1) See [M2] Theorem 11.5. (2) Let $a \in \mathfrak{p}$ be any non-zero element, and write $a = p_1 \cdots p_r$ as a product of prime elements. Since $\mathfrak{p}$ is a prime ideal, there exists i with $p_i \in \mathfrak{p}$, so $0 \neq (p_i) \subseteq \mathfrak{p}$, and since $\mathfrak{p}$ is a minimal non-zero prime ideal, we obtain $\mathfrak{p} = (p_i)$. (See [M2] Theorem 20.1 for a converse.) $\square$

Example 3.24. Let k be a field, $A = k[T_1, \dots, T_n]$ the polynomial ring in $n > 1$ variables, an integrally closed (even factorial) domain. Let $X = \mathbb{A}_k^n = \text{Spec}(A)$ and $U = \mathbb{A}_k^n \setminus \{0\}$. Then (inside $K(\mathbb{A}_k^n) = \text{Frac}(A)$) we have

$$\Gamma(X, \mathcal{O}_X) \subseteq \Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_{X,x} \subseteq \bigcap_{x \in X, \dim(\mathcal{O}_{X,x})=1} \mathcal{O}_x = \Gamma(X, \mathcal{O}_X),$$

where the final equality follows from Part (1) of the lemma. This shows that $\Gamma(U, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$. (In the case at hand, this can alternatively be shown by an elementary computation with polynomials.) In particular U is not affine. The same argument works, if $U \subset X$ is the complement of any finite non-empty set of closed points.

Proposition 3.25. *Let X be a normal integral noetherian scheme.*

- (1) *The map cyc is injective and induces an injective map $\text{DivCl}(X) \rightarrow \text{Cl}(X)$.*
- (2) *If X is locally factorial, then the map cyc is a group isomorphism $\text{Div}(X) \cong Z^1(X)$ and induces an isomorphism $\text{DivCl}(X) \cong \text{Cl}(X)$.*

Sketch of proof. Injectivity. If D is a Weil divisor or a Cartier divisor such that D and $-D$ are effective, then D is trivial. It therefore suffices to show that the inverse image of the subset of effective Weil divisors under the homomorphism cyc consists of effective Cartier divisors. So let D be a Cartier divisor on X such that $\text{cyc}(D)$ is effective. We can check that D is effective locally on X , so we may assume that $X = \text{Spec } A$ for an integrally closed domain A , and that D is principal, say given by (X, f) . By assumption $f \in K(X)$ is contained in $A_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spec } A$ of height 1, and it follows from Lemma 3.23 Part (1) that $f \in A$, as desired.

Surjectivity. Now assume that X is locally factorial. We construct an inverse to the map cyc . If $Z \subset X$ is an integral closed subscheme of X of codimension 1 with corresponding ideal sheaf $\mathcal{I}_Z \subseteq \mathcal{O}_X$, then for every $x \in X$, $\mathcal{I}_{Z,x}$ is a principal ideal in $\mathcal{O}_{X,x}$ by Lemma 3.23 Part (2). Using Proposition 2.19, we find an affine open cover $(U_i)_i$ of X together with elements $f_i \in K(X)$ such that $\mathcal{I}_{Z|U_i} = f_i \mathcal{O}_X$ (inside $\mathcal{K}_X$), for each i . We then map $[Z]$ to the Cartier divisor $(U_i, f_i)_i$ (well-defined since elements in a domain that generate the same principal ideal differ at most by a unit), and extend this construction to a map $d: Z^1(X) \rightarrow \text{Div}(X)$ by linearity. By construction we have $\text{cyc} \circ d = \text{id}$, and this implies that cyc is surjective (and hence bijective with inverse d).

We can phrase the definition of principal Weil divisor as saying that it is the image under cyc of a principal Cartier divisor. It is therefore clear that

we also obtain an injection (or an isomorphism, respectively, if X is locally factorial) $\text{DivCl}(X) \rightarrow \text{Cl}(X)$. $\square$

For X locally factorial we thus have identifications

$$\text{Pic}(X) \cong \text{DivCl}(X) \cong \text{Cl}(X).$$

Example 3.26.

- (1) For any UFD A , $\text{Pic}(A) = 1$ as remarked above. In particular, all divisors on affine space $\mathbb{A}_k^n$ over a field (or over any UFD) k are principal.
- (2) Let k be a field. As shown on the problem sheets, $\text{Pic}(\mathbb{P}_k^1) \cong \mathbb{Z}$. We will see below that $\text{Pic}(\mathbb{P}_k^n) \cong \mathbb{Z}$ for every $n \geq 1$.

The Picard group or equivalently the divisor class group of an integral scheme X contains interesting information about X , but is often not easy to compute.

(3.7) Basics on algebraic curves.

Let k be an algebraically closed field.

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Definition 3.27. An integral normal proper curve over k is an integral proper k -scheme which is a Dedekind scheme of dimension 1.

Remark 3.28.

- (1) One can show that every (integral normal) proper curve X over k is a projective k -scheme, i.e., there exist $n \geq 1$ and a closed immersion $X \hookrightarrow \mathbb{P}_k^n$. Conversely, as we have seen, the existence of such an immersion implies that X is proper over k . (The assumptions that X is integral and normal simplify the proof a little, but are not required for the result.) See Problem 24 or [GW1] Theorem 15.18.
- (2) With the notion of *smooth morphism of schemes* (which we have not yet discussed) one can show that for every integral normal (proper) curve X over k the structure morphism $X \rightarrow \text{Spec}(k)$ is smooth (in other words, X is a smooth k -scheme). Here it is essential that k is algebraically closed.

In view of (1), (2), an integral normal proper curve over k is the same as an *integral smooth projective curve over k* (and in the literature probably the latter term is used more frequently).

A prototypical example (albeit not the most interesting one) is the projective line $\mathbb{P}_k^1$.

To each integral scheme, in particular to each integral normal proper curve X over k we can attach its field $K(X)$ of rational functions. In the case at hand, $K(X)$ is a finitely generated extension field of k (being the field of fractions of a finitely generated k -algebra) of transcendence degree $\text{trdeg}(K(X)/k) = \dim(X) = 1$.

The closed subsets of an integral normal proper curve X are $\emptyset$, X and finite sets of closed points. (In fact, every closed subset has only finitely

many irreducible components, and since $\dim(X) = 1$, the only *irreducible* closed subsets are X and the sets consisting of a single closed point.

For normal proper curves, the field of rational functions “contains the full information about the curve” in the sense of the following theorem. We start with a lemma that clarifies which class of morphisms we consider.

Lemma 3.29. *Let $f: X \rightarrow Y$ be a morphism between integral normal proper curves over k . Denote by $\eta_X \in X$, $\eta_Y \in Y$ the generic points. The following are equivalent.*

- (i) *f is surjective,*
- (ii) *f is dominant, i.e., $\overline{f(X)} = Y$, equivalently $f(\eta_X) = \eta_Y$,*
- (iii) *f is non-constant (i.e., f does not factor through a morphism $\text{Spec}(k) \rightarrow Y$),*
- (iv) *all fibers of f are finite.*

Proof. Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). To show (iii) $\Rightarrow$ (iv) first note that the fiber over η_Y is either empty or equal to $\{\eta_X\}$. (In fact, a closed point $x \in X$ cannot map to η_Y , because we would then obtain chain of inclusion $k = \kappa(x) \supseteq \kappa(\eta_Y) = K(Y) \supseteq k$, contradicting the fact that the transcendence degree of $K(Y)$ over k is 1.) Fibers over closed points are closed, so either all fibers are finite or there exists a closed point $y \in Y$ with $f^{-1}(y) = X$, i.e., f is constant. We also see that f is dominant, if all fibers are finite, because then η_X cannot be mapped to a closed point in Y . This shows (iv) $\Rightarrow$ (ii). Finally, (ii) $\Rightarrow$ (i) holds, because f is proper (Lemma 1.23). $\square$

Theorem 3.30. *Mapping a curve X to its field $K(X)$ of rational functions defines an anti-equivalence of categories between*

- *the category of integral normal proper curves over the algebraically closed field k with dominant k -scheme morphisms as morphisms, and*
- *the category of finitely generated extension fields K/k of transcendence degree 1.*

Proof. To prove that the functor is fully faithful, let X, Y be integral normal proper curves over k and consider the map

$$\text{Hom}_k^{\text{dom}}(X, Y) \rightarrow \text{Hom}_k(K(Y), K(X))$$

from the set of dominant k -morphisms $X \rightarrow Y$ to the set of k -algebra homomorphisms $K(Y) \rightarrow K(X)$. We split up the proof into the following two lemmas (the first for injectivity, the second for surjectivity) where we can relax the assumptions a little.

We prove the essential surjectivity later on. $\square$

Lemma 3.31. *Let X and Y be integral k -schemes, and assume that Y is separated over k . Then the map $\text{Hom}_k^{\text{dom}}(X, Y) \rightarrow \text{Hom}_k(K(Y), K(X))$ is injective.*

Proof. This holds, because Y is separated over k and $\text{Spec}(K(X))$ has dense image in X . (Given two morphisms $f, g: X \rightarrow Y$, their equalizer is a closed subscheme in X , because Y is separated over k ; if f and g induce the same morphism on function fields, then the equalizer contains the generic point of X , hence equals the integral scheme X .) $\square$

Lemma 3.32. *Let X and Y be integral k -schemes, and assume that X is a Dedekind scheme and that Y is proper over k . Then the map $\text{Hom}_k^{\text{dom}}(X, Y) \rightarrow \text{Hom}_k(K(Y), K(X))$ is surjective.*

Proof. Let $\varphi: K(Y) \rightarrow K(X)$ be a k -homomorphism. It is enough to find, for every closed point x in X , an open neighborhood U_x and a morphism $U_x \rightarrow Y$ which induces φ on function fields. In fact, the previous lemma shows that these morphisms necessarily coincide on the intersections $U_x \cap U_{x'}$, and we obtain the desired morphism $X \rightarrow Y$ by gluing.

So let $x \in X$ be a closed point. By the following lemma we find a morphism $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow Y$ which induces φ . Its image is contained in some affine open $V \subset Y$ (any open neighborhood of the image of the closed point will do), so we have a ring homomorphism $\psi: \Gamma(V, \mathcal{O}_Y) \rightarrow \mathcal{O}_{X,x}$. Let U be an affine open neighborhood of x . Then x corresponds to a prime ideal $\mathfrak{p} \subset \Gamma(U, \mathcal{O}_X)$ and $\mathcal{O}_{X,x}$ is the localization $\Gamma(U, \mathcal{O}_X)_{\mathfrak{p}}$. Since $\Gamma(V, \mathcal{O}_Y)$ is a finitely generated k -algebra, the image of ψ is contained in a suitable localization of $\Gamma(U, \mathcal{O}_X)$ with respect to one element not contained in the prime ideal corresponding to x , i.e., we may extend the morphism $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow Y$ to some principal open subset of U that still contains x . This concludes the proof of surjectivity. $\square$

Lemma 3.33. *Let $f: Y \rightarrow S$ be a proper morphism, and let R be a discrete valuation ring with field of fractions K . Assume that we have a commutative diagram*

$$\begin{array}{ccc} \text{Spec}(K) & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ \text{Spec}(R) & \longrightarrow & S, \end{array}$$

where the left vertical arrow is the natural open embedding.

Then there exists a unique arrow $\text{Spec}(R) \rightarrow Y$ making the resulting diagram commutative.

Proof. The uniqueness is clear since f is separated. To show existence, we can equivalently show that the morphism $f': Y \times_S \text{Spec}(R) \rightarrow \text{Spec}(R)$ has a section $s: \text{Spec}(R) \rightarrow Y \times_S \text{Spec}(R)$, i.e., $f' \circ s = \text{id}$, that extends the morphism $\text{Spec}(K) \rightarrow Y \times_S \text{Spec}(R)$ given by the diagram. Let y be the image point of $\text{Spec}(K)$. We view the closure $\overline{\{y\}}$ as a reduced closed subscheme of $Y \times_S \text{Spec}(R)$. Since f is proper, f' is closed, thus $f'(\overline{\{y\}}) = \text{Spec}(R)$. So there exists $y' \in \overline{\{y\}}$ mapping to the closed point of

$\text{Spec}(R)$. We then obtain a chain of inclusions

$$R \subseteq \mathcal{O}_{\overline{\{y\}}, y'} \subseteq \mathcal{O}_{\overline{\{y\}}, y} \subseteq K.$$

Since $\mathcal{O}_{\overline{\{y\}}, y}$ is of dimension 0 and reduced, it is a field. Since it contains R , we have $\mathcal{O}_{\overline{\{y\}}, y} = K$. We claim that $R = \mathcal{O}_{\overline{\{y\}}, y'}$; this immediately gives us the desired section. Now, since R is a valuation ring, for $s \in \mathcal{O}_{\overline{\{y\}}, y'}$, if $s \notin R$, then $s^{-1} \in \mathfrak{m}_R$. But then $s^{-1} \in \mathcal{O}_{\overline{\{y\}}, y'}^\times$, contradicting the fact that the inclusion $R \subseteq \mathcal{O}_{\overline{\{y\}}, y'}$ is a local ring homomorphism. $\square$

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Remark 3.34.

- (1) The valuative criterion for properness ([GW1] Theorem 15.9) says that a morphism of finite type between noetherian schemes is proper if and only if for every discrete valuation ring R , every diagram as above can be filled in with a unique diagonal arrow as in the lemma. (Similarly, replacing *unique* by *at most one*, one obtains a valuative criterion for a morphism being separated.)
- (2) For $S = \text{Spec}(k)$, $Y = \mathbb{P}_k^1$ (and similarly for $\mathbb{P}_k^n$) it is easy to prove the lemma directly, in other words to check that $\mathbb{P}_k^1$ satisfies the valuative criterion for properness. In fact, let R, K with a diagram as in the lemma be given. Let v be the valuation for R , and let $t \in R$ be a uniformizer, i.e., $v(t) = 1$. First note that since R is a local ring, any morphism $\text{Spec}(R) \rightarrow \mathbb{P}_k^1$ factors through the open embedding of one of the standard charts. Therefore we have

$$\mathbb{P}_k^1(R) = \{(x_0 : x_1); x_i \in R, \text{ at least one of } x_0, x_1 \text{ is in } R^\times\},$$

where as usual we consider $(x_0 : x_1) = (ux_0 : ux_1)$ for $u \in R^\times$. Now given $(x_0 : x_1) \in \mathbb{P}^1(K)$ we set $m = \min(v(x_0), v(x_1))$. Then we have $(x_0 : x_1) = (t^{-m}x_0 : t^{-m}x_1) \in \mathbb{P}^1(R)$, and this gives exactly the (unique) R -valued point that we wanted to construct.

More explicitly, write X_0, X_1 for the homogeneous coordinates on $\mathbb{P}_k^1$. Say $m = \min(v(x_0), v(x_1)) = v(x_0)$, so that $t^{-m}x_0 \in R^\times$ and $t^{-m}x_1 \in R$. Then the morphism $\text{Spec}(R) \rightarrow \mathbb{P}_k^1$ factors as

$$\text{Spec}(R) \rightarrow \text{Spec } k \left[\frac{X_1}{X_0} \right] = D_+(X_0) \hookrightarrow \mathbb{P}_k^1,$$

where the first morphism is given by $\frac{X_1}{X_0} \mapsto \frac{x_1}{x_0} = \frac{t^{-m}x_1}{t^{-m}x_0} \in R$.

Recall that a morphism $f: X \rightarrow Y$ of schemes is called *finite*, if for every affine open $V \subseteq Y$ the inverse image $f^{-1}(V)$ is affine and the ring homomorphism $\Gamma(V, \mathcal{O}_Y) \rightarrow \Gamma(f^{-1}(V), \mathcal{O}_X)$ is finite. This can be checked on an affine cover of Y (Problem 9, [GW1] Proposition 12.9). Every finite morphism is proper (using a little commutative algebra about integral ring homomorphisms (“going up”) this is not difficult; see [GW1] Examples 12.56 (3)).

It is also true that every finite morphism with target a projective k -scheme is itself projective (this is more difficult, see [GW1] Corollary 13.77; the instance used below, namely that for every finite morphism $f: X \rightarrow \mathbb{P}_k^1$ of k -schemes, X is projective over k , is easier to prove and we may come back to that later; of course this also follows from the above-mentioned fact that every proper curve over k is projective).

We can now sketch the proof that the functor $X \rightarrow K(X)$ is essentially surjective.

Proof of essential surjectivity in Theorem 3.30. Let K/k be a finitely generated field extension of transcendence degree 1. Let $t \in K \setminus k$ be any element. Since k is algebraically closed, t is transcendental over k . Since the transcendence degree is 1, the extension $K/k(t)$ is algebraic, and since K/k is finitely generated, it is even finite.

We want to construct an integral smooth projective curve X , such that $K(X) = K$. (The embedding $K(\mathbb{P}_k^1) = k(t) \rightarrow K$ will then correspond to a morphism $X \rightarrow \mathbb{P}_k^1$; this points us, in some sense, in the right direction to construct X .)

We define X as the “normalization of $\mathbb{P}_k^1$ in K ”. We sketch the construction here and refer to [GW1] Sections (12.11), (15.7) for further details. For any affine open $U \subset \mathbb{P}_k^1$, say $U = \text{Spec}(A)$ with $A \subset k(t) \subseteq K$, let $\tilde{A}$ be the integral closure of A in K . This is a Dedekind ring and the ring homomorphism $A \rightarrow \tilde{A}$ is finite ([GW1] Corollary 12.52 (1)). Set $\tilde{U} = \text{Spec}(\tilde{A})$.

Since taking integral closure in some extension field is compatible with localization, this construction “globalizes”, i.e., we can glue all the schemes $\tilde{U}$ for varying U to obtain a scheme X . Since X is covered by the $\tilde{U}$, it is an integral Dedekind scheme. Moreover, by construction the morphism $X \rightarrow \mathbb{P}_k^1$ is finite, hence X is proper over k . $\square$

A morphism $f: X \rightarrow Y$ is called *flat*, if for every $x \in X$ the ring homomorphism $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is flat (equivalently, the ring homomorphisms $\Gamma(V, \mathcal{O}_Y) \rightarrow \Gamma(U, \mathcal{O}_X)$ for all affine open subschemes $V \subseteq Y$ and $U \subseteq f^{-1}(V)$ are flat).

Lemma 3.35. *Let $f: X \rightarrow Y$ be a dominant morphism between integral normal proper curves over k . Then f is finite and flat. Therefore, for every affine open $V \subset Y$, the inverse image $f^{-1}(V)$ is affine and $\Gamma(f^{-1}(V), \mathcal{O}_X)$ is a locally free $\Gamma(V, \mathcal{O}_Y)$ -module of rank $\deg(f) := [K(X) : K(Y)]$.*

Proof. For flatness, note that the homomorphisms between the local rings are injective, because those are subrings of the respective fields of rational functions. Since a module over a discrete valuation ring is flat if and only if it is torsion-free, we get the result.

To show that f is finite, choose an element $t \in K(Y)$ that is transcendental over k . Its image in $K(X)$ of course is again transcendental over k . By the

construction used in the proof of essential surjectivity in Theorem 3.30, we obtain morphisms $X \rightarrow Y \rightarrow \mathbb{P}_k^1$ such that $Y \rightarrow \mathbb{P}_k^1$ is finite, in particular separated, and the composition $X \rightarrow \mathbb{P}_k^1$ is finite. It follows by Lemma 1.23 (1) that $X \rightarrow Y$ is finite.

For the final statement, we use Theorem 2.23. $\square$

In particular, every morphism $f: X \rightarrow Y$ of degree 1 is an isomorphism: Use either the fully faithful functor to pass to function fields; or use the fact that every ring homomorphism $A \rightarrow B$ such that B is a locally free A -module of rank 1 necessarily is an isomorphism (in fact, we may check this Zariski-locally on $\text{Spec}(A)$, so may assume that B is free of rank 1; but then the claim is clear).

Let X be an integral normal proper curve. In view of Proposition 3.25, we identify Weil and Cartier divisors on X . A Weil divisor is a finite formal linear combination $\sum n_P [P]$ of closed (equivalently: k -valued) points on X with integer coefficients.

Definition 3.36. For a (Weil) divisor $D = \sum_P n_P [P]$ we define the degree $\deg(D)$ of D as $\deg(D) := \sum_P n_P$. We obtain a group homomorphism $Z^1(X) \rightarrow \mathbb{Z}$.

Definition 3.37. Let $f: X \rightarrow Y$ be a dominant morphism of integral normal proper curves, and let D be a divisor on Y . We define the pullback $f^*(D)$ as the following divisor.

We represent $D = (V_i, g_i)_i$ as a Cartier divisor and define $f^*(D) = (f^{-1}(V_i), \varphi(g_i))_i$, where $\varphi: K(Y) \rightarrow K(X)$ is the injection induced by f .

Equivalently, from the perspective of Weil divisors, we describe the pullback $f^*([y])$ for a closed point y (and extend the definition by linearity). In that special case, we obtain $f^*([y]) = \sum_{x, f(x)=y} v_x(f_x^\sharp(t))[x]$, where $f_x^\sharp: \mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X, x}$ is the morphism between local rings, v_x is the valuation for the discrete valuation ring $\mathcal{O}_{X, x}$ and $t \in \mathcal{O}_{Y, f(x)} = \mathcal{O}_{Y, y}$ is a uniformizer.

To check the equivalence (which holds for any flat morphism between finite type k -schemes that are Dedekind schemes of dimension 1) we may work locally and assume that $D = [y]$, that $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ are affine and that there exists $t \in A$ with $\text{div}(t) = D$ (i.e., t is a uniformizer in A_y and a unit in all other localizations of A at maximal ideals). Then $f^*([y]) = \text{div}(\varphi(t))$ clearly has the two equivalent descriptions above.

Proposition 3.38. Let $f: X \rightarrow Y$ be a dominant morphism of integral normal proper curves, and let D be a divisor on Y . We have

$$\deg(f^*(D)) = \deg(f) \cdot \deg(D).$$

Proof. It is enough to handle the case $D = [y]$ for some closed point $y \in Y$. We need to show that $\deg(f^*([y])) = \deg(f) = [K(X) : K(Y)]$.

Claim. We have $\deg(f^*([y])) = \dim_k \Gamma(f^{-1}(y), \mathcal{O}_{f^{-1}(y)})$, where $f^{-1}(y) = X \times_Y \operatorname{Spec}(\kappa(y))$ is the scheme-theoretic fiber.

Once we have proved the claim, the lemma follows, because we have $\dim_k \Gamma(f^{-1}(y), \mathcal{O}_{f^{-1}(y)}) = [K(X) : K(Y)]$, since f is finite flat (this number is the rank of the locally free module $\Gamma(f^{-1}(V), \mathcal{O}_X)$ over $\Gamma(V, \mathcal{O}_Y)$, $V \subset Y$ affine open).

Proof of claim. Let $V = \operatorname{Spec}(A)$ be an affine open neighborhood of y such that there exists $t \in A$ with $V(t) = \{y\}$ as a reduced closed subscheme. (Extend a uniformizer of $\mathcal{O}_{Y,y}$ to a suitable open neighborhood.) Write $f^{-1}(V) = \operatorname{Spec}(B)$, where B is a finite A -algebra. There exist only finitely many (closed) points in X mapping to y , say $x_1, \dots, x_r$. We have

$$\Gamma(f^{-1}(y), \mathcal{O}_{f^{-1}(y)}) = B \otimes_A A/t = B/tB = \prod_{i=1}^r \mathcal{O}_{X,x_i}/(t),$$

The final equality holds because $\operatorname{Spec}(B/tB)$ topologically is the finite space $\{x_1, \dots, x_r\}$ with the discrete topology, so the ring B/tB is the product of the finitely many local rings at its points. We have $\dim_k(\mathcal{O}_{X,x_i}/(t)) = v_{x_i}(t)$. The claim follows. $\square$

For an element $f \in K(X)^\times$ that is transcendental over k , we obtain a homomorphism $K(\mathbb{P}_k^1) = k(t) \rightarrow K(X)$ mapping t to f , and in view of the above theorem a finite flat morphism $\bar{f}: X \rightarrow \mathbb{P}_k^1$. Then the principal divisor attached to f is the “configuration of zeros and poles of $\bar{f}$ ” in the following precise sense.

Lemma 3.39. *With notation as above, $\operatorname{div}(f) = \bar{f}^*([0] - [\infty])$.*

Proof. Identifying $K(\mathbb{P}_k^1) = k(t)$ implies that we choose a 0 and ∞ in $\mathbb{P}_k^1$. In other words, the standard charts of $\mathbb{P}_k^1$ are $\operatorname{Spec}(k[t])$ (and 0 corresponds to the prime ideal (t) in this chart) and $\operatorname{Spec}(k[t^{-1}])$ (and ∞ corresponds to the prime ideal (t^{-1}) in this chart).

We have $\bar{f}(x) = 0$ if and only if the morphism $\operatorname{Spec}(\mathcal{O}_{X,x}) \rightarrow \mathbb{P}_k^1$ induced by $\bar{f}$ factors through $\operatorname{Spec}(k[t])$, i.e., $f \in \mathcal{O}_{X,x}$, and maps the maximal ideal $\mathfrak{m}_x$ to (t) , i.e., $v_x(f) > 0$. In this case $v_x(f)$ is the coefficient of $\operatorname{div}(f)$ at x , and also the coefficient of $\bar{f}^*([0])$ at x .

Symmetrically, we have an analogous picture for ∞ in place of 0. Altogether, the claim follows. $\square$

We can now show that the degree homomorphism $Z^1(X) \rightarrow \mathbb{Z}$ factors through $\operatorname{Cl}(X)$:

Theorem 3.40. *Let $f \in K(X)^\times$. Then $\deg(\operatorname{div}(f)) = 0$.*

Proof. By the lemma, we have $\operatorname{div}(f) = \bar{f}^*([0] - [\infty])$, and hence

$$\deg(\operatorname{div}(f)) = \deg(\bar{f}) \cdot \deg([0] - [\infty]) = 0.$$

$\square$

In particular, we can speak of the degree of a line bundle, and we denote the degree of $\mathcal{L}$ by $\deg(\mathcal{L})$.

Proposition 3.41. *Let X be an integral normal proper curve over k . If there exist closed points $x \neq y$ on X such that $[x] \sim [y]$, then $X \cong \mathbb{P}_k^1$.*

Proof. By assumption, $[x] - [y] = \operatorname{div}(f)$ for some $f \in K(X)^\times$. Since $x \neq y$, the divisor $[x] - [y]$ is non-trivial, so f is transcendental over k and defines a dominant morphism $f: X \rightarrow \mathbb{P}_k^1$. Then $f^*[0] = [x]$, so f has degree 1 by Proposition 3.38 and therefore is an isomorphism. $\square$

Proposition 3.42. *Let X be an integral normal proper curve over k . If there exists a closed point x on X such that $\dim \Gamma(X, \mathcal{O}([x])) > 1$, then $X \cong \mathbb{P}_k^1$.*

Proof. See Problem 22. $\square$

(3.8) The theorem of Riemann and Roch.

No proofs were given in the lecture at this point for the following results.

Reference: [H] IV.1, [GW1] Sections (15.9), (15.10), [GW2] (26.11).

Let X be an integral normal proper curve over an algebraically closed field k .

To state the famous Theorem of Riemann–Roch, we introduce the following notation. For a divisor D we write $\ell(D) = \dim_k \Gamma(X, \mathcal{O}_X(D))$, where

$$\Gamma(X, \mathcal{O}_X(D)) = \{f \in K(X)^\times; \operatorname{div}(f) + D \geq 0\} \cup \{0\},$$

as before.

Remark 3.43.

- (1) We have $\ell(D) > 0$ if and only if D is linearly equivalent to an effective divisor (in fact, the divisors linearly equivalent to D are exactly those of the form $\operatorname{div}(f) + D$). In particular $\deg(D) < 0$ implies $\ell(D) = 0$.
- (2) We have $\ell(0) = \dim_k \Gamma(X, \mathcal{O}_X) = 1$ since $\Gamma(X, \mathcal{O}_X) = k$ (Problem 22).

Proposition 3.44. *For each D , $\ell(D)$ is finite.*

Theorem 3.45. (Riemann–Roch) *For X as above, there exist $g \in \mathbb{Z}_{\geq 0}$ and $K \in \operatorname{Div}(X)$ such that for every divisor D on X , we have*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

Corollary 3.46. *In the above situation, we have*

- (1) $\ell(K) = g$,
- (2) $\deg(K) = 2g - 2$,
- (3) for every D with $\deg(D) > 2g - 2$, we have $\ell(D) = \deg(D) + 1 - g$.

Proof. The corollary is easy to prove with the Theorem of Riemann-Roch at hand. In fact, for (1) use the theorem with $D = 0$ the trivial divisor, for (2) use $D = K$, and for (3) use that under the assumption there $\deg(K - D) < 0$, whence $\ell(K - D) = 0$, as remarked before. $\square$

The number g is called the *genus* of the curve X . Part (3) of the corollary shows that it is uniquely determined by X .

Remark 3.47. The linear equivalence class of the canonical divisor K is uniquely determined. In fact, assume that K and K' are divisors which both have the property of a canonical divisor as in the Riemann-Roch theorem. Using the theorem and the corollary, one computes that $\ell(K - K') > 0$ and $\ell(K' - K) > 0$. As was shown on the problem sheet, this implies that $\mathcal{O}_X(K - K')$ is trivial, or in other words that K and K' have the same divisor class.

For the projective line $\mathbb{P}_k^1$, it is easy to prove the Theorem of Riemann-Roch by direct computations. It has genus 0. One can show that every X of genus 0 is isomorphic to $\mathbb{P}_k^1$. (But if k is not assumed to be algebraically closed, then there may exist X as above of genus 0 which are not isomorphic to $\mathbb{P}_k^1$.)

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For X as above which is of the form $V_+(f) \subset \mathbb{P}_k^2$, there is the following formula for the genus:

Proposition 3.48. *Let X as above be of the form $V_+(f) \subset \mathbb{P}_k^2$ for a homogeneous polynomial f of degree d . Then the genus g of X is given by*

$$g = \frac{(d-1)(d-2)}{2}.$$

For example, elliptic curves (which are defined by a homogeneous polynomial of degree 3) have genus 1.

(3.9) Line bundles on $\mathbb{P}_k^n$.

References: [GW1], Ch. 8, Ch. 11, in particular Example 11.43, (8.5); [H] II.6, II.7.

We want to compute the Picard group of projective space over a field. To this end, we will use the following general proposition.

Proposition 3.49. *Let X be a noetherian integral scheme such that all local rings $\mathcal{O}_{X,x}$ are unique factorization domains. Let $U \subseteq X$ be an open subscheme, and let $Z_1, \dots, Z_r$ be those irreducible components of $X \setminus U$ that are of codimension 1 inside X . We consider the Z_i as integral closed subschemes of X . Then we have a short exact sequence*

$$0 \rightarrow \bigoplus_i \mathbb{Z}[Z_i] \rightarrow Z^1(X) \rightarrow Z^1(U) \rightarrow 0$$

which induces an exact sequence

$$\bigoplus_i \mathbb{Z}[Z_i] \rightarrow \mathrm{Cl}(X) \rightarrow \mathrm{Cl}(U) \rightarrow 0.$$

Proof. It is clear that we have the first short exact sequence when we think of integral closed subschemes in terms of their generic points. It is easy to check that the “restriction map” $Z^1(X) \rightarrow Z^1(U)$ induces a homomorphism between the class groups, and this yields the second exact sequence. $\square$

In terms of the identifications of the divisor class groups with the Picard groups of X and of U , the map $\mathrm{Pic}(X) \rightarrow \mathrm{Pic}(U)$ in the proposition is just the restriction of line bundles from X to the open subscheme U .

Now let R be a ring and fix $n \geq 1$. We cover $\mathbb{P}_R^n$ by the standard charts $U_i := D_+(X_i)$, as usual, and write $U_{ij} := U_i \cap U_j$. For $d \in \mathbb{Z}$, multiplication by the elements $(X_i/X_j)^d \in \Gamma(U_{ij}, \mathcal{O}_{\mathbb{P}_R^n}^\times)$ defines isomorphisms $\mathcal{O}_{U_i|U_{ij}} \rightarrow \mathcal{O}_{U_j|U_{ij}}$ which give rise to a gluing datum of the $\mathcal{O}_{U_i}$ -modules $\mathcal{O}_{U_i}$. By gluing of sheaves, we obtain a line bundle $\mathcal{O}_{\mathbb{P}_R^n}(d)$. (Cf. Problems 12, 13, 17 in the case $n = 1$.) To shorten the notation, we sometimes just write $\mathcal{O}(d)$, when the space is clear from the context.

Lemma 3.50. *We obtain a group homomorphism $\mathbb{Z} \rightarrow \mathrm{Pic}(\mathbb{P}_R^n)$, $d \mapsto \mathcal{O}(d)$.*

Proposition 3.51. *Writing $R[X_0, \dots, X_n]_d$ for the submodule of homogeneous polynomials of degree d (with $R[X_0, \dots, X_n]_d = 0$ for $d < 0$), we have natural isomorphisms*

$$\Gamma(\mathbb{P}_R^n, \mathcal{O}(d)) \cong R[X_0, \dots, X_n]_d$$

for all $d \in \mathbb{Z}$.

Proof. We can make the gluing construction for $\mathcal{O}(d)$ explicit by identifying $\Gamma(D_+(X_i), \mathcal{O}(d))$ with $X_i^d R \left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i} \right]$, and correspondingly

$$\Gamma(D_+(X_i X_j), \mathcal{O}(d)) = X_i^d R \left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}, \frac{X_i}{X_j} \right] = X_j^d R \left[\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}, \frac{X_j}{X_i} \right]$$

(inside $R[X_0, \dots, X_n, X_0^{-1}, \dots, X_n^{-1}]$). This means that the restriction map is just the inclusion map. (If R is a domain, then this describes an embedding of $\mathcal{O}(d)$ into the constant sheaf $\mathcal{K}_{\mathbb{P}_R^n}$ with values the field of rational functions.)

With this description, since the relevant restriction maps are injective, we can identify $\Gamma(\mathbb{P}_R^n, \mathcal{O}(d))$ with the intersection

$$\bigcap_{i=0}^n X_i^d R \left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i} \right].$$

One checks that this intersection is $R[X_0, \dots, X_n]_d$, as claimed. $\square$

Corollary 3.52. *The above homomorphism $\mathbb{Z} \rightarrow \mathrm{Pic}(\mathbb{P}_R^n)$, $d \mapsto \mathcal{O}(d)$, is injective.*

Now let $R = k$ be a field (in fact, the same arguments apply to any noetherian unique factorization domain k). Then $\mathbb{P}_k^n$ is a noetherian integral scheme all of whose local rings are unique factorization domains, so we can talk about Cartier divisors and about Weil divisors, and identify the two notions via the cycle map as in Proposition 3.25.

Corollary 3.53. *Let k be a field and let $n \geq 1$. Then $\text{Pic}(\mathbb{P}_k^n) \cong \mathbb{Z}$.*

Proof. If we identify $\text{Cl}(\mathbb{P}_k^n) = \text{Pic}(\mathbb{P}_k^n)$ and apply Proposition 3.49 to $X = \mathbb{P}_k^n$ and $U = D_+(X_0) \cong \mathbb{A}_k^n$, we obtain a surjection $\mathbb{Z} \rightarrow \text{Pic}(\mathbb{P}_k^n)$. The injectivity statement of the previous corollary implies that this surjection (which allows us to identify $\text{Pic}(\mathbb{P}_k^n)$ with some quotient of $\mathbb{Z}$) must be an isomorphism. (At this point we have not yet shown that the map $d \mapsto \mathcal{O}(d)$ is surjective, and hence an isomorphism; this will follow from the discussion in the following section.) $\square$

Remark 3.54. One can show that every locally free $\mathcal{O}_{\mathbb{P}_k^1}$ -module is isomorphic to a direct sum of line bundles. Note though that this statement is not true for $\mathbb{P}_k^n$, $n > 1$.

(3.10) Divisors on $\mathbb{P}_k^n$.

Let k be a field (or more generally a noetherian unique factorization domain), and let $n \geq 1$. Let us take a look at the line bundles $\mathcal{O}(d)$ from the point of view of Cartier divisors. Write

$$\mathcal{R} = \{f = \frac{g}{h}; g, h \in k[X_0, \dots, X_n] \text{ non-zero homogeneous polynomials}\}.$$

Then $\mathcal{R}$ is a group with respect to multiplication. For $f = g/h \in \mathcal{R}$, we define $\deg(f) = \deg(g) - \deg(h)$. We can identify $K(X)^\times$ with the subgroup of $\mathcal{R}$ of degree 0 elements.

Fix an element $f \in \mathcal{R}$ and write $d = \deg(f)$. Let D be the Cartier divisor $\text{div}(f) := (D_+(X_i), f/X_i^d)_i$. (This is a new use of the symbol div since f is not an element of $K(\mathbb{P}_k^n)$; but when $\deg(f) = 0$, then f may be considered as an element of $K(\mathbb{P}_k^n)$ and the two definitions of $\text{div}(f)$ coincide.). Describing the line bundle $\mathcal{O}_{\mathbb{P}_k^n}(D)$ in terms of a gluing datum, it follows that $\mathcal{O}_X(D) \cong \mathcal{O}(d)$.

Proposition 3.55. *The map $\mathcal{R} \rightarrow \text{Div}(\mathbb{P}_k^n)$, $f \mapsto \text{div}(f)$, is a surjective group homomorphism. The composition*

$$\mathcal{R} \rightarrow \text{Div}(\mathbb{P}_k^n) \rightarrow \text{Pic}(\mathbb{P}_k^n) \cong \mathbb{Z}$$

is the degree map on $\mathcal{R}$. In particular, the isomorphism class of the line bundle $\mathcal{O}_{\mathbb{P}_k^n}(\text{div}(f))$ depends only on d , not on the choice of f , and the map $\mathbb{Z} \rightarrow \text{Pic}(\mathbb{P}_k^n)$, $d \mapsto \mathcal{O}_{\mathbb{P}_k^n}(d)$, is an isomorphism.

Proof. It is easy to see that $f \mapsto \text{div}(f)$ is a group homomorphism. Since $\text{DivCl}(\mathbb{A}_k^n) = 0$, every Cartier divisor on $\mathbb{P}_k^n$ has a representative of the form

$(D_+(X_i), f_i)$ for suitable $f_i \in K(\mathbb{P}_k^n)^\times$. (In fact, given D , we may restrict D to $D_+(X_i)$ where it becomes principal, say $D|_{D_+(X_i)} = (D_+(X_i), f_i)$.) Set $f := X_0^d f_0$. Then it is easy to check that $D = \div(f)$ as Cartier divisors. This shows the surjectivity of the map $\mathcal{R} \rightarrow \text{Div}(\mathbb{P}_k^n)$.

All the other assertions follow from this. (Recall that we have already shown that $\mathbb{Z} \rightarrow \text{Pic}(\mathbb{P}_k^n)$, $d \mapsto \mathcal{O}_{\mathbb{P}_k^n}(d)$, is injective.) $\square$

Now let $f \in k[X_0, \dots, X_n]$ be an irreducible homogeneous polynomial of degree $d > 0$. Then $V_+(f)$ is an integral closed subscheme of $\mathbb{P}_k^n$ of codimension 1 (since the same is true after intersection with any of the open charts $D_+(X_i)$ (unless the intersection is empty)). From the construction of the matching between Cartier and Weil divisors, one sees that the Cartier divisor $\text{div}(f)$ defined above corresponds to the Weil divisor $[V_+(f)]$. As a particular example, for any fixed i , the Weil divisor $[V_+(X_i)]$ of the line $V_+(X_i)$ has associated line bundle $\mathcal{O}(1)$.

Since the identification of Cartier divisors with Weil divisors is a group isomorphism, one can extend this description to all divisors, by decomposing a general $f \in \mathcal{R}$ as a product of irreducible homogeneous polynomials and of inverses of such polynomials.

Let again f be a homogeneous polynomial of degree $d > 0$. The datum of the divisor $\text{div}(f)$ corresponds to the choice of an embedding of its associated line bundle $\mathcal{O}(d)$ into $\mathcal{K}_{\mathbb{P}_k^n}$. The image of this embedding contains the structure sheaf $\mathcal{O}_{\mathbb{P}_k^n}$, and going through the definitions shows that the global section $1 \in \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n})$ is mapped to $f \in k[X_0, \dots, X_n]_d = \Gamma(\mathbb{P}_k^n, \mathcal{O}(d))$ (Proposition 3.51) under this embedding. In other words, the embedding $\Gamma(\mathbb{P}_k^n, \mathcal{O}(d)) \rightarrow K(X)$ of the global sections is given by

$$k[X_0, \dots, X_n]_d \rightarrow K(X), \quad g \mapsto g/f.$$

Corollary 3.56. *Let k be a field, and let $Z \subseteq \mathbb{P}_k^n$ be an integral closed subscheme of codimension 1. Then $Z = V_+(f)$ for some homogeneous polynomial f .*

Proof. Consider the divisor $[Z]$ given by Z . Viewed as a Cartier divisor, it corresponds to an embedding $\mathcal{O}_{\mathbb{P}_k^n}([Z]) \rightarrow \mathcal{K}_{\mathbb{P}_k^n}$ whose image contains $\mathcal{O}_{\mathbb{P}_k^n}$ since the divisor $[Z]$ is effective. Let $f \in k[X_0, \dots, X_n]_d$ be the image of $1 \in \Gamma(\mathbb{P}_k^n, \mathcal{O})$ in $\Gamma(\mathbb{P}_k^n, \mathcal{O}([Z])) = k[X_0, \dots, X_n]_d$ under this embedding. Since the embedding $\mathcal{O}_{\mathbb{P}_k^n}([Z]) \rightarrow \mathcal{K}_{\mathbb{P}_k^n}$ is entirely determined by this image, the above discussion shows that $[Z] = [V_+(f)]$ as divisors and hence that $Z = V_+(f)$. $\square$

(3.11) Functorial description of $\mathbb{P}^n$.

As we have seen in Section 1.1, every scheme X defines a contravariant functor $T \mapsto X(T) := \text{Hom}_{(\text{Sch})}(T, X)$ from the category of schemes to the category of sets. This functor determines X up to unique isomorphism.

In this section, we want to describe the functor attached in this way to projective space $\mathbb{P}_R^n$ for R a ring.

Lemma 3.57. *Let X be a scheme.*

- (1) *Let $\mathcal{F}$ be an $\mathcal{O}_X$ -module. Giving an $\mathcal{O}_X$ -module homomorphism $\alpha: \mathcal{O}_X^{n+1} \rightarrow \mathcal{F}$ is “the same” as giving global sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{F})$ (namely the images of the standard basis vectors of $\Gamma(X, \mathcal{O}_X^{n+1}) = \Gamma(X, \mathcal{O}_X)^{n+1}$).*
- (2) *Now let $\mathcal{L}$ be a line bundle on X , and let $\alpha: \mathcal{O}_X^{n+1} \rightarrow \mathcal{L}$ be an $\mathcal{O}_X$ -module homomorphism given by $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Then α is surjective if and only if for every $x \in X$ there exists i such that $s_i(x) \neq 0$ in the fiber $\mathcal{L}(x)$.*

Proposition 3.58. *Let R be a ring, and let S be an R -scheme. There are bijections, functorial in S ,*

$$\mathbb{P}_R^n(S) = \{(\mathcal{L}, \alpha); \mathcal{L} \text{ a line bundle on } S, \\ \alpha: \mathcal{O}_S^{n+1} \rightarrow \mathcal{L} \text{ a surjective } \mathcal{O}_S\text{-module homom.}\} / \cong.$$

Here we consider pairs $(\mathcal{L}, \alpha)$, $(\mathcal{L}', \alpha')$ as isomorphic, if there exists an $\mathcal{O}_S$ -module isomorphism $\beta: \mathcal{L} \rightarrow \mathcal{L}'$ with $\alpha = \alpha' \circ \beta$.

Saying that the bijections of the proposition are functorial means that given a morphism $S' \rightarrow S$ of R -schemes the bijections for S and S' together with the natural map $\mathbb{P}_R^n(S) \rightarrow \mathbb{P}_R^n(S')$ and the map $(\mathcal{L}, \alpha) \mapsto (g^*\mathcal{L}, g^*\alpha)$ give rise to a commutative diagram. (Note that for every line bundle $\mathcal{L}$ on S the pull-back $g^*\mathcal{L}$ is a line bundle on S' , and for surjective α the pull-back $g^*\alpha$ is again a surjective $\mathcal{O}_{S'}$ -homomorphism of the desired form.)

Proof. A homomorphism $\alpha: \mathcal{O}_S^{n+1} \rightarrow \mathcal{L}$ corresponds to $n+1$ global sections in $\Gamma(S, \mathcal{L})$ (the “images of the standard basis vectors”). Thus $X_0, \dots, X_n \in \Gamma(\mathbb{P}_R^n, \mathcal{O}(1))$ give rise to a homomorphism $\mathcal{O}_{\mathbb{P}_R^n}^{n+1} \rightarrow \mathcal{O}(1)$. This homomorphism is surjective. (In fact, looking back at the construction of $\mathcal{O}(1)$ by gluing and the way how we identified the global sections of $\mathcal{O}(1)$ with $R[X_0, \dots, X_n]_1$, under the identification $\mathcal{O}(1)_{D_+(X_i)} \cong \mathcal{O}_{D_+(X_i)}$ the restriction of X_i to $D_+(X_i)$ corresponds to $1 \in \Gamma(D_+(X_i), \mathcal{O}_{D_+(X_i)})$ and in particular is non-zero in every fiber.)

Given a morphism $S \rightarrow \mathbb{P}_R^n$, we can pull this homomorphism back to S and obtain an element of the right hand side in the statement of the proposition.

Conversely, given a pair $(\mathcal{L}, \alpha)$ on S , we can think of the corresponding morphism $S \rightarrow \mathbb{P}_R^n$ in terms of homogeneous coordinates (i.e., for K -valued points for some field K), as follows: Denote by $f_0, \dots, f_n \in \Gamma(S, \mathcal{L})$ the global sections corresponding to α . For a point $x \in S$, the fiber $\mathcal{L}(x)$ is a one-dimensional $\kappa(x)$ -vector space generated by the elements $f_0(x), \dots, f_n(x)$ (i.e., at least one of them is $\neq 0$ – this holds since α is surjective). We choose an isomorphism $\mathcal{L}(x) \cong \kappa(x)$, and hence can view the $f_i(x)$ as elements of $\kappa(x)$. Then the morphism $S \rightarrow \mathbb{P}_R^n$ maps x to $(f_0(x) : \dots : f_n(x)) \in \mathbb{P}^n(\kappa(x))$. While the individual $f_i(x)$, as elements of $\kappa(x)$, depend on the choice of

isomorphism $\mathcal{L}(x) \cong \kappa(x)$, the point $(f_0(x) : \cdots : f_n(x)) \in \mathbb{P}^n(\kappa(x))$ is independent of this choice.

To make this rigorous, consider a pair $(\mathcal{L}, \alpha)$ as above, and for $i \in \{0, \dots, n\}$ define

$$S_i = \{s \in S; \alpha(e_i) \text{ generates the fiber } \mathcal{L}(s)\},$$

where $e_i \in \mathcal{O}_S^{n+1}$ denotes the i -th standard basis vector. This defines an open cover of S . By definition, composing α with the injection $\mathcal{O}_S \rightarrow \mathcal{O}_S^{n+1}$ as the i -th summand induces a trivialization $\mathcal{O}_{S_i} \cong \mathcal{L}|_{S_i}$ of the restriction of $\mathcal{L}$. We obtain a morphism $S_i \rightarrow D_+(X_i) = \operatorname{Spec} k[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}]$ by mapping $\frac{X_j}{X_i}$ to the image of $\alpha(e_j) \in \Gamma(S_i, \mathcal{L})$ under the isomorphism $\Gamma(S_i, \mathcal{L}) \rightarrow \Gamma(S_i, \mathcal{O}_{S_i})$. These morphisms can be glued, and one obtains the desired morphism $S \rightarrow \mathbb{P}_R^n$.

To conclude the proof, one checks that the two constructions are inverse to each other. $\square$

Remark 3.59. If $S = \operatorname{Spec} k$ for a field k , then every locally free sheaf on S is free, and the proposition reads as

$$\mathbb{P}^n(k) = \{\alpha: k^{n+1} \rightarrow k \text{ surjective}\} / \cong,$$

where now homomorphisms α, α' are isomorphic if and only if they have the same kernel. Thus we can identify

$$\mathbb{P}^n(k) = \{U \subset k^{n+1} \text{ sub-vector space of dimension } n\}.$$

This description is dual to the classical description of $\mathbb{P}^n(k)$ as the set of lines in k^{n+1} . Passing to the dual space, the projection $k^{n+1} \rightarrow k^{n+1}/U$ induces an inclusion $(k^{n+1}/U)^\vee \rightarrow k^{n+1,\vee}$ of the dual vector spaces. Matching the standard basis of k^{n+1} with its dual basis, we can identify $k^{n+1,\vee} = k^{n+1}$, and in this way we get back the description in terms of lines.

Let X be any integral scheme. We can think of effective Cartier divisors as closed subschemes, in the following sense.

Proposition 3.60. *Let X be an integral scheme. There are bijections between*

- (i) *The set of effective Cartier divisors on X ,*
- (ii) *the set of pairs $(\mathcal{L}, s)$, where $\mathcal{L}$ is a locally free $\mathcal{O}_X$ -module of rank 1 and $s \in \Gamma(X, \mathcal{L}) \setminus \{0\}$, up to isomorphism,*
- (iii) *the set of closed subschemes $Z \subset X$ that locally on X are defined by a single equation, i.e., denoting by $\mathcal{I}$ the ideal sheaf of Z , for every $x \in X$ there exists $f \in \mathcal{O}_{X,x} \setminus \{0\}$ such that $\mathcal{I}_x = (f) \subseteq \mathcal{O}_{X,x}$,*

given by mapping an effective Cartier divisor D to $(\mathcal{O}_X(D), 1)$ (where $1 \in \Gamma(X, \mathcal{O}_X(D))$) via the inclusions $\mathcal{O}_X \subseteq \mathcal{O}_X(D) \subset \mathcal{K}_X$), and by mapping $(\mathcal{L}, s)$ to the ideal sheaf $\mathcal{L}^\vee \subseteq \mathcal{O}_X$, where the inclusion is the dual of the homomorphism $s: \mathcal{O}_X \rightarrow \mathcal{L}$.

In (ii), we call $(\mathcal{L}, s)$ and $(\mathcal{L}', s')$ isomorphic if there is an isomorphism $\mathcal{L} \xrightarrow{\cong} \mathcal{L}'$ of $\mathcal{O}_X$ -modules which maps s to s' .

Proof. It is not difficult to construct inverse maps for the maps described in the statement of the proposition. $\square$

Remark 3.61. Let k be a field. Let X be an integral k -scheme and let $\iota: X \rightarrow \mathbb{P}_k^n$ be a closed immersion, corresponding to a line bundle $\mathcal{L}$ on X together with global sections $s_0, \dots, s_n$. Assume that $s_i \neq 0$ for all i (geometrically, this means that the image of X is not contained in any of the hyperplanes $V_+(X_i)$). Then the effective Cartier divisors D_i corresponding to the pairs $(\mathcal{L}, s_i)$ in the sense of the previous definition are exactly the closed subschemes $\iota^{-1}(V_+(X_i))$ of X , when viewed as elements of the set described in Part (iii) of the proposition. Thus the closed embedding ι is determined by the $n + 1$ effective Cartier divisors on X (all linearly equivalent to each other) that are the intersections of X with the coordinate hyperplanes.

Of course, an arbitrary collection of $n + 1$ linearly equivalent effective Cartier divisors may not define a morphism (and even if it does, will usually not define a closed immersion) from X to $\mathbb{P}_k^n$. If $\mathcal{L}$ is a locally free $\mathcal{O}_X$ -module of rank 1 such that $\Gamma(X, \mathcal{L})$ has a basis of global sections defining a closed immersion into projective space, $\mathcal{L}$ is called *very ample*. We will come back to this notion later (but maybe not in the current term.)

The following remark was not discussed in class, but maybe should have been ...

Remark 3.62. Let k be an algebraically closed field, let X be a k -scheme of finite type of dimension $\dim X > 0$, and let $\iota: X \rightarrow \mathbb{P}_k^n$ be a morphism, corresponding to a line bundle $\mathcal{L}$ on X . If ι is injective (e.g., if ι is a closed immersion), then for any two closed points $x, x' \in X$, there must exist a global section s of $\mathcal{L}$ (even inside the span of the subspace generated by those global sections of $\mathcal{L}$ that define ι) with $s(x) = 0$, $s(x') \neq 0$. (Use a suitable change of coordinates.) Taking tensor powers of $\mathcal{L}$ and the sections, we see that every $\mathcal{L}^{\otimes d}$, $d \geq 1$, has the same property. In particular, $\mathcal{L}^{\otimes d}$ is a non-trivial line bundle (i.e., not isomorphic to $\mathcal{O}_X$).

Now assume that X is integral and that ι is a closed immersion. Then Problem 20 shows that $\Gamma(X, \mathcal{L}^{\otimes d, \vee}) = 0$ for every $d \geq 1$.

Let Z be an effective Cartier divisor in $\mathbb{P}_k^n$, say $\mathcal{O}(Z) \cong \mathcal{O}(d)$. Assume that Z is not linearly equivalent to the trivial divisor, i.e., $d > 0$. Then $X \cap Z$ is non-empty. (This is a result that goes in the direction of Bézout's theorem.) In fact, if $X \subseteq Z$, then $X \cap Z = X \neq \emptyset$.

Now assume that $X \not\subseteq Z$. Then the schematic intersection $X \cap Z := X \times_{\mathbb{P}_k^n} Z$ is an effective Cartier divisor on X (locally defined by the image of the equation defining Z in $\mathcal{O}_X$) with topological space the set-theoretic intersection $X \cap Z$. Its line bundle is $\mathcal{L}^d$, which is non-trivial by the previous discussion. Thus the intersection cannot be empty.

4. SMOOTHNESS AND DIFFERENTIALS *

General reference: [GW1] Ch. 6.

The Zariski tangent space.

(4.1) Definition of the Zariski tangent space.

Definition 4.1. Let X be a scheme, $x \in X$, $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ the maximal ideal in the local ring at x , $\kappa(x)$ the residue class field of X in x . The $\kappa(x)$ -vector space $(\mathfrak{m}/\mathfrak{m}^2)^*$ is called the (Zariski) tangent space of X in x .

Definition 4.2. Let R be a ring, $f_1, \dots, f_r \in R[T_1, \dots, T_n]$. We call the matrix

$$J_{f_1, \dots, f_r} := \left(\frac{\partial f_i}{\partial T_j} \right)_{i,j} \in M_{r \times n}(R[T_\bullet])$$

the Jacobian matrix of the polynomials f_i . Here the partial derivatives are to be understood in a formal sense.

Remark 4.3.

- (1) If in the above setting the ideal $\mathfrak{m}$ is finitely generated, then $\dim_{\kappa(x)} T_x X$ is the minimal number of elements needed to generate $\mathfrak{m}$ and in particular is finite.
- (2) The tangent space construction is functorial in the following sense: Given a scheme morphism $f: X \rightarrow Y$ and $x \in X$ such that $\dim_{\kappa(x)} T_x X$ is finite or $[\kappa(x) : \kappa(f(x))]$ is finite, then we obtain a map

$$df_x: T_x X \rightarrow T_{f(x)} Y \otimes_{\kappa(f(x))} \kappa(x).$$

Example 4.4. Let k be a field, $X = V(f_1, \dots, f_m) \subseteq \mathbb{A}_k^n$, $f_i \in k[T_1, \dots, T_n]$, $x = (x_i)_i \in k^n = \mathbb{A}^n(k)$. Then there is a natural identification $T_x X = \text{Ker}(J_{f_1, \dots, f_m}(x))$, where $J_{f_1, \dots, f_m}(x)$ denotes the matrix with entries in $\kappa(x) = k$ obtained by mapping each entry of $J_{f_1, \dots, f_m}$ to $\kappa(x)$, which amounts to evaluating these polynomials at x .

Proposition 4.5. Let k be a field, X a k -scheme, $x \in X(k)$. There is an identification (functorial in (X, x))

$$X(k[\varepsilon]/(\varepsilon^2))_x := \{f \in \text{Hom}_k(\text{Spec } k[\varepsilon]/(\varepsilon^2), X); \text{im}(f) = \{x\}\} = T_x X.$$

Smooth morphisms.

(4.2) Definition of smooth morphisms.

Definition 4.6. A morphism $f: X \rightarrow Y$ of schemes is called smooth of relative dimension $d \geq 0$ in $x \in X$, if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \operatorname{Spec} R \subseteq Y$ of $f(x)$ such that $f(U) \subseteq V$ and an open immersion $j: U \rightarrow \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d})$ such that the triangle

$$\begin{array}{ccc} U & \xrightarrow{j} & \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d}) \\ & \searrow f & \swarrow \\ & V & \end{array}$$

is commutative, and that the Jacobian matrix $J_{f_1, \dots, f_{n-d}}(x)$ has rank $n - d$.

We say that $f: X \rightarrow Y$ is smooth of relative dimension d if f is smooth of relative dimension d at every point of X . Instead of smooth of relative dimension 0, we also use the term *étale*.

With notation as above, if f is smooth at $x \in X$, then x has an open neighborhood such that f is smooth at all points of this open neighborhood. Clearly, $\mathbb{A}_S^n$ and $\mathbb{P}_S^n$ are smooth of relative dimension n for every scheme S . (It is harder to give examples of non-smooth schemes directly from the definition; we will come back to this later.)

(4.3) Dimension of schemes.

Recall from commutative algebra that for a ring R we define the (Krull) dimension $\dim R$ of R as the supremum over all lengths of chains of prime ideals, or equivalently as the dimension of the topological space $\operatorname{Spec} R$ in the sense of the following definition.

Definition 4.7. Let X be a topological space. We define the dimension of X as

$$\dim X := \sup\{\ell; \text{there exists a chain } Z_0 \supsetneq Z_1 \supsetneq \dots \supsetneq Z_\ell \\ \text{of closed irreducible subsets } Z_i \subseteq X\}.$$

We will use this notion of dimension for non-affine schemes, as well. Recall the following theorem about the dimension of finitely generated algebras over a field from commutative algebra:

Theorem 4.8. Let k be a field, and let A be a finitely generated k -algebra which is a domain. Let $\mathfrak{m} \subset A$ be a maximal ideal. Then

$$\dim A = \operatorname{trdeg}_k(\operatorname{Frac}(A)) = \dim A_{\mathfrak{m}}.$$

By passing to an affine cover, we obtain the following corollary:

Corollary 4.9. Let k be a field, and let X be an integral k -scheme which is of finite type over k . Denote by $K(X)$ its field of rational functions. Let $U \subseteq X$ be a non-empty open subset, and let $x \in X$ be a closed point. Then

$$\dim X = \dim U = \operatorname{trdeg}_k(K(X)) = \dim \mathcal{O}_{X,x}.$$

(4.4) Existence of smooth points.

Let k be a field.

Lemma 4.10. *Let X, Y be [integral²] k -schemes which are locally of finite type over k . Let $x \in X, y \in Y$, and let $\varphi: \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ be an isomorphism of k -algebras. Then there exist open neighborhoods U of x and V of y and an isomorphism $h: U \rightarrow V$ of k -schemes with $h_x^\# = \varphi$.*

Proposition 4.11. *Let X be an integral k -scheme of finite type. Assume that $K(X) \cong k(T_1, \dots, T_d)[\alpha]$ with α separable algebraic over $k(T_1, \dots, T_d)$. (This is always possible if k is perfect.) (Then $\dim X = d$ by the above.)*

Then there exists a dense open subset $U \subseteq X$ and a separable irreducible polynomial $g \in k(T_1, \dots, T_d)[T]$ with coefficients in $k[T_1, \dots, T_d]$, such that U is isomorphic to a dense open subset of $\operatorname{Spec} k[T_1, \dots, T_d]/(g)$.

Theorem 4.12. *Let k be a perfect field, and let X be a nonempty reduced k -scheme locally of finite type over k . Then the smooth locus*

$$X_{\text{sm}} := \{x \in X; X \rightarrow \operatorname{Spec} k \text{ is smooth in } x\}$$

of X is open and dense.

(4.5) Regular rings.

For references to the literature, see [GW1] App. B, in particular B.73, B.74, B.75

Definition 4.13. *A noetherian local ring A with maximal ideal $\mathfrak{m}$ and residue class field κ is called regular, if $\dim A = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.*

One can show that the inequality $\dim A \leq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ always holds. Therefore we can rephrase the definition as saying that A is regular if $\mathfrak{m}$ has a generating system consisting of $\dim A$ elements.

Definition 4.14. *A noetherian ring A is called regular, if $A_{\mathfrak{m}}$ is regular for every maximal ideal $\mathfrak{m} \subset A$.*

We quote the following (mostly non-trivial) results about regular rings:

Theorem 4.15.

- (1) *Every localization of a regular ring is regular.*
- (2) *If A is regular, then the polynomial ring $A[T]$ is regular.*
- (3) *(Theorem of Auslander–Buchsbaum) Every regular local ring is factorial.*
- (4) *Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and of dimension d , and let $f_1, \dots, f_r \in \mathfrak{m}$. Then $A/(f_1, \dots, f_r)$ is regular of dimension $d - r$ if and only if the images of the f_i in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $A/\mathfrak{m}$.*

²The statement is true in general, but in the lecture we proved it only with the additional assumption that X and Y are integral.

(4.6) Smoothness and regularity.

Let k be a field.

Lemma 4.16. *Let X be a k -scheme locally of finite type. Let $x \in X$ such that $X \rightarrow \operatorname{Spec} k$ is smooth of relative dimension d in x . Then $\mathcal{O}_{X,x}$ is regular of dimension $\leq d$. If moreover x is closed, then $\mathcal{O}_{X,x}$ is regular of dimension d .*

Lemma 4.17. *Let $X = V(g_1, \dots, g_s) \subseteq \mathbb{A}_k^n$, and let $x \in X$ be a closed point. If $\operatorname{rk} J_{g_1, \dots, g_s}(x) \geq n - \dim \mathcal{O}_{X,x}$, then x is smooth in X/k , and $\operatorname{rk} J_{g_1, \dots, g_s}(x) = n - \dim \mathcal{O}_{X,x}$.*

Theorem 4.18. *Let X be a k -scheme locally of finite type, $x \in X$ a closed point, $d \geq 0$. Fix an algebraically closed extension field K of k and write $X_K = X \otimes_k K$. The following are equivalent:*

- (i) *The morphism $X \rightarrow \operatorname{Spec} k$ is smooth of relative dimension d at x .*
- (ii) *For all points $\bar{x} \in X_K$ lying over x , X_K is smooth over K of relative dimension d at $\bar{x}$.*
- (iii) *There exists a point $\bar{x} \in X_K$ lying over x , such that X_K is smooth over K of relative dimension d at $\bar{x}$.*
- (iv) *For all points $\bar{x} \in X_K$ lying over x , the local ring $\mathcal{O}_{X_K, \bar{x}}$ is regular of dimension d .*
- (v) *There exists a point $\bar{x} \in X_K$ lying over x , such that the local ring $\mathcal{O}_{X_K, \bar{x}}$ is regular of dimension d .*

If these conditions hold, then the local ring $\mathcal{O}_{X,x}$ is regular of dimension d , and if $\kappa(x) = k$, then this last condition is equivalent to the previous ones.

Corollary 4.19. *Let X be an irreducible scheme of finite type over k , and let $x \in X(k)$ be a k -valued point. Then $X \rightarrow \operatorname{Spec} k$ is smooth at x if and only if $\dim X = \dim_k T_x X$.*

Corollary 4.20. *Let $X = V(g_1, \dots, g_s) \subseteq \mathbb{A}_k^n$ and let $x \in X$ be a smooth closed point. Let $d = \dim \mathcal{O}_{X,x}$. Then $J_{g_1, \dots, g_s}(x)$ has rank $n - d$. In particular, $s \geq n - d$.*

After renumbering the g_i , if necessary, there exists an open neighborhood U of x and an open immersion $U \subseteq V(g_1, \dots, g_{n-d})$, i.e., locally around x , “ X is cut out in affine space by the expected number of equations”.

Corollary 4.21. *Let X be locally of finite type over k . The following are equivalent:*

- (i) *X is smooth over k .*
- (ii) *$X \otimes_k L$ is regular for every field extension L/k .*
- (iii) *There exists an algebraically closed extension field K of k such that $X \otimes_k K$ is regular.*

The sheaf of differentials.

General references: [M2] §25, [Bo] Ch. 8, [H] II.8.

(4.7) Modules of differentials.

Let A be a ring.

Definition 4.22. *Let B be an A -algebra, and M a B -module. An A -derivation from B to M is a homomorphism $D: B \rightarrow M$ of abelian groups such that*

- (a) *(Leibniz rule) $D(bb') = bD(b') + b'D(b)$ for all $b, b' \in B$,*
- (b) *$d(a) = 0$ for all $a \in A$.*

Assuming property (a), property (b) is equivalent to saying that D is a homomorphism of A -modules. We denote the set of A -derivations $B \rightarrow M$ by $\text{Der}_A(B, M)$; it is naturally a B -module.

Definition 4.23. *Let B be an A -algebra. We call a B -module $\Omega_{B/A}$ together with an A -derivation $d_{B/A}: B \rightarrow \Omega_{B/A}$ a module of (relative, Kähler) differentials of B over A if it satisfies the following universal property:*

For every B -module M and every A -derivation $D: B \rightarrow M$, there exists a unique B -module homomorphism $\psi: \Omega_{B/A} \rightarrow M$ such that $D = \psi \circ d_{B/A}$.

In other words, the map $\text{Hom}_B(\Omega_{B/A}, M) \rightarrow \text{Der}_A(B, M)$, $\psi \mapsto \psi \circ d_{B/A}$ is a bijection.

Lemma 4.24. *Let I be a set, $B = A[T_i, i \in I]$ the polynomial ring. Then $\Omega_{B/A} := B^{(I)}$ with $d_{B/A}(T_i) = e_i$, the “ i -th standard basis vector” is a module of differentials of B/A .*

So we can write $\Omega_{B/A} = \bigoplus_{i \in I} B d_{B/A}(T_i)$.

Lemma 4.25. *Let $\varphi: B \rightarrow B'$ be a surjective homomorphism of A -algebras, and write $\mathfrak{b} = \text{Ker}(\varphi)$. Assume that a module of differentials $(\Omega_{B/A}, d_{B/A})$ for B/A exists. Then*

$$\Omega_{B/A}/(\mathfrak{b}\Omega_{B/A} + B'd(\mathfrak{b}))$$

together with the derivation $d_{B'/A}$ induced by $d_{B/A}$ is a module of differentials for B'/A .

Corollary 4.26. *For every A -algebra B , a module $\Omega_{B/A}$ of differentials exists. It is unique up to unique isomorphism.*

We will see later that for a scheme morphism $X \rightarrow Y$, one can construct an $\mathcal{O}_X$ -module $\Omega_{X/Y}$ together with a “derivation” $\mathcal{O}_X \rightarrow \Omega_{X/Y}$ by gluing sheaves associated to modules of differentials attached to the coordinate rings of suitable affine open subschemes of X and Y .

Let $\varphi: A \rightarrow B$ be a ring homomorphism. For the next definition, we will consider the following situation: Let C be a ring, $I \subseteq C$ an ideal with $I^2 = 0$, and let

$$\begin{array}{ccc} B & \longrightarrow & C/I \\ \uparrow \varphi & & \uparrow \\ A & \longrightarrow & C \end{array}$$

be a commutative diagram (where the right vertical arrow is the canonical projection). We will consider the question whether for these data, there exists a homomorphism $B \rightarrow C$ (dashed in the following diagram) making the whole diagram commutative:

$$\begin{array}{ccc} B & \longrightarrow & C/I \\ \uparrow \varphi & \dashrightarrow & \uparrow \\ A & \longrightarrow & C \end{array}$$

Definition 4.27. Let $\varphi: A \rightarrow B$ be a ring homomorphism.

- (1) We say that φ is *formally unramified*, if in every situation as above, there exists at most one homomorphism $B \rightarrow C$ making the diagram commutative.
- (2) We say that φ is *formally smooth*, if in every situation as above, there exists at least one homomorphism $B \rightarrow C$ making the diagram commutative.
- (3) We say that φ is *formally étale*, if in every situation as above, there exists a unique homomorphism $B \rightarrow C$ making the diagram commutative.

Passing to the spectra of these rings, we can interpret the situation in geometric terms: $\text{Spec } C/I$ is a closed subscheme of $\text{Spec } C$ with the same topological space, so we can view the latter as an “infinitesimal thickening” of the former. The question becomes the question whether we can extend the morphism from $\text{Spec } C/I$ to $\text{Spec } B$ to a morphism from this thickening.

Proposition 4.28. Let $\varphi: A \rightarrow B$ be a ring homomorphism. Then φ is formally unramified if and only if $\Omega_{B/A} = 0$.

For an algebraic field extension L/K one can show that $K \rightarrow L$ is formally unramified if and only if it is formally smooth if and only if L/K is separable. Cf. Problem 27 and [M2] §25, §26 (where the discussion is extended to the general, not necessarily algebraic, case).

Theorem 4.29. Let $f: A \rightarrow B$, $g: B \rightarrow C$ be ring homomorphisms. Then we obtain a natural sequence of C -modules

$$\Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A} \rightarrow \Omega_{C/B} \rightarrow 0$$

which is exact.

If moreover g is formally smooth, then the sequence

$$0 \rightarrow \Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A} \rightarrow \Omega_{C/B} \rightarrow 0$$

is a split short exact sequence.

Theorem 4.30. *Let $f: A \rightarrow B$, $g: B \rightarrow C$ be ring homomorphisms. Assume that g is surjective with kernel $\mathfrak{b}$. Then we obtain a natural sequence of C -modules*

$$\mathfrak{b}/\mathfrak{b}^2 \rightarrow \Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A} \rightarrow 0,$$

where the homomorphism $\mathfrak{b}/\mathfrak{b}^2 \rightarrow \Omega_{B/A} \otimes_B C$ is given by $x \mapsto d_{B/A}(x) \otimes 1$.

If moreover $g \circ f$ is formally smooth, then the sequence

$$0 \rightarrow \mathfrak{b}/\mathfrak{b}^2 \rightarrow \Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A} \rightarrow 0$$

is a split short exact sequence.

(4.8) The sheaf of differentials of a scheme morphism.

Remark 4.31. Let again B an A -algebra. There is the following alternative construction of $\Omega_{B/A}$: Let $m: B \otimes_A B \rightarrow B$ be the multiplication map, and let $I = \text{Ker}(m)$. Then I/I^2 is a B -module, and $d: B \rightarrow I/I^2, b \mapsto 1 \otimes b - b \otimes 1$, is an A -derivation. One shows that $(I/I^2, d)$ satisfies the universal property defining $(\Omega_{B/A}, d_{B/A})$.

We can use a similar definition as we used for ring homomorphisms above to define the notions of formally unramified, formally smooth and formally étale morphisms of schemes.

Definition 4.32. *Let $f: X \rightarrow Y$ be a morphism of schemes.*

- (1) *We say that f is formally unramified, if for every ring C , every ideal I with $I^2 = 0$, and every morphism $\text{Spec } C \rightarrow Y$ (which we use to view $\text{Spec } C$ and $\text{Spec } C/I$ as Y -schemes), the composition with the natural closed embedding $\text{Spec } C/I \rightarrow \text{Spec } C$ yields an injective map $\text{Hom}_Y(\text{Spec } C, X) \rightarrow \text{Hom}_Y(\text{Spec } C/I, X)$.*
- (2) *We say that f is formally smooth, if for every ring C , every ideal I with $I^2 = 0$, and every morphism $\text{Spec } C \rightarrow Y$, the composition with the natural closed embedding $\text{Spec } C/I \rightarrow \text{Spec } C$ yields a surjective map $\text{Hom}_Y(\text{Spec } C, X) \rightarrow \text{Hom}_Y(\text{Spec } C/I, X)$.*
- (3) *We say that f is formally étale, if f is formally unramified and formally smooth.*

If f is a morphism of affine schemes, then f has one of the properties of this definition if and only if the corresponding ring homomorphism has the same property in the sense of our previous definition.

Lemma 4.33.

- (1) *Every monomorphism of schemes (in particular: every immersion) is formally unramified.*
- (2) *Let $A \rightarrow B \rightarrow C$ be ring homomorphisms such that $A \rightarrow B$ is formally unramified. Then we can naturally identify $\Omega_{C/A} = \Omega_{C/B}$.*

Definition 4.34. Let $X \rightarrow Y$ be a morphism of schemes, and let $\mathcal{M}$ be an $\mathcal{O}_X$ -module. A derivation $D: \mathcal{O}_X \rightarrow \mathcal{M}$ is a homomorphism of abelian sheaves such that for all open subsets $U \subseteq X$, $V \subseteq Y$ with $f(U) \subseteq V$, the map $\mathcal{O}(U) \rightarrow \mathcal{M}(U)$ is an $\mathcal{O}_Y(V)$ -derivation.

Equivalently, $D: \mathcal{O}_X \rightarrow \mathcal{M}$ is a homomorphism of $f^{-1}(\mathcal{O}_Y)$ -modules such that for every open $U \subseteq X$, the Leibniz rule

$$D(U)(bb') = bD(U)(b') + b'D(U)(b), \quad \forall b, b' \in \Gamma(U, \mathcal{O}_X)$$

holds.

We denote the set of all these derivations by $\text{Der}_Y(\mathcal{O}_X, \mathcal{M})$; it is a $\Gamma(X, \mathcal{O}_X)$ -module.

Definition/Proposition 4.35. Let $f: X \rightarrow Y$ be a morphism of schemes. The following three definitions give the same result (up to unique isomorphism), called the sheaf of differentials of f or of X over Y , denoted $\Omega_{X/Y}$ — a quasi-coherent $\mathcal{O}_X$ -module together with a derivation $d_{X/Y}: \mathcal{O}_X \rightarrow \Omega_{X/Y}$.

- (i) There exists a unique $\mathcal{O}_X$ -module $\Omega_{X/Y}$ together with a derivation $d_{X/Y}: \mathcal{O}_X \rightarrow \Omega_{X/Y}$ such that for all affine open subsets $\text{Spec } B = U \subseteq X$, $\text{Spec } A = V \subseteq Y$ with $f(U) \subseteq V$, $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$ and $d_{X/Y|U}$ is induced by $d_{B/A}$.
- (ii) $\Omega_{X/Y} = \Delta^*(\mathcal{I} / \mathcal{I}^2)$, where $\Delta: X \rightarrow X \times_Y X$ is the diagonal morphism, $W \subseteq X \times_Y X$ is open such that $\text{im}(\Delta) \subseteq W$ is closed (if f is separated we can take $W = X \times_Y X$), and $\mathcal{I}$ is the quasi-coherent ideal defining the closed subscheme $\Delta(X) \subseteq W$. The derivation $d_{X/Y}$ is induced, on affine opens, by the map $b \mapsto 1 \otimes b - b \otimes 1$.
- (iii) The quasi-coherent $\mathcal{O}_X$ -module $\Omega_{X/Y}$ together with $d_{X/Y}$ is characterized by the universal property that composition with $d_{X/Y}$ induces bijections

$$\text{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{M}) \xrightarrow{\sim} \text{Der}_Y(\mathcal{O}_X, \mathcal{M})$$

for every quasi-coherent $\mathcal{O}_X$ -module $\mathcal{M}$, functorially in $\mathcal{M}$.

The properties we proved for modules of differentials can be translated into statements for sheaves of differentials:

Proposition 4.36. Let $f: X \rightarrow Y$, $g: Y' \rightarrow Y$ be morphisms of schemes, and let $X' = X \times_Y Y'$. Denote by $g': X' \rightarrow X$ the base change of g . There is a natural isomorphism $\Omega_{X'/Y'} \cong (g')^* \Omega_{X/Y}$, compatible with the universal derivations.

Proposition 4.37. Let $f: X \rightarrow Y$, $g: Y \rightarrow Z$ be morphisms of schemes. Then there is an exact sequence

$$f^* \Omega_{Y/Z} \rightarrow \Omega_{X/Z} \rightarrow \Omega_{X/Y} \rightarrow 0$$

of $\mathcal{O}_X$ -modules. If f is formally smooth, then the sequence

$$0 \rightarrow f^* \Omega_{Y/Z} \rightarrow \Omega_{X/Z} \rightarrow \Omega_{X/Y} \rightarrow 0$$

is exact and splits locally on X .

Proposition 4.38. *Let $i: Z \rightarrow X$ be a closed immersion with corresponding ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, and let $g: X \rightarrow Y$ be a scheme morphism. Then there is an exact sequence*

$$i^*(\mathcal{I} / \mathcal{I}^2) \rightarrow i^*\Omega_{X/Y} \rightarrow \Omega_{Z/Y} \rightarrow 0$$

of $\mathcal{O}_Z$ -modules. If Z is formally smooth over Y , then the sequence

$$0 \rightarrow i^*(\mathcal{I} / \mathcal{I}^2) \rightarrow i^*\Omega_{X/Y} \rightarrow \Omega_{Z/Y} \rightarrow 0$$

is exact and splits locally on Z .

Proposition 4.39. *Let K be a field, and let X be a k -scheme of finite type. Let $x \in X(k)$. Then we have an isomorphism $T_x X = \Omega_{X/k}(x)$ between the Zariski tangent space at x and the fiber of the sheaf of differentials of X/k at x .*

Proposition 4.40. *Let R be a ring. We have a short exact sequence*

$$0 \rightarrow \Omega_{\mathbb{P}^n_R/R} \rightarrow \mathcal{O}(-1)^{n+1} \rightarrow \mathcal{O}^n \rightarrow 0$$

of $\mathcal{O}_X$ -modules.

(4.9) Sheaves of differentials and smoothness.

We start by slightly rephrasing the definition of a smooth morphism.

Definition 4.41. *A morphism $f: X \rightarrow Y$ of schemes is called smooth of relative dimension $d \geq 0$ in $x \in X$, if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \text{Spec } R \subseteq Y$ of $f(x)$ such that $f(U) \subseteq V$ and an open immersion $j: U \rightarrow \text{Spec } R[T_1, \dots, T_n](f_1, \dots, f_{n-d})$ such that the triangle*

$$\begin{array}{ccc} U & \xrightarrow{j} & \text{Spec } R[T_1, \dots, T_n](f_1, \dots, f_{n-d}) \\ & \searrow f & \swarrow \\ & V & \end{array}$$

is commutative, and that the images of $df_1, \dots, df_{n-d}$ in the fiber $\Omega_{\mathbb{A}_R^n/R}^1 \otimes \kappa(x)$ are linearly independent over $\kappa(x)$. (We view x as a point of $\mathbb{A}_R^n$ via the embedding $U \rightarrow \text{Spec } R[T_1, \dots, T_n](f_1, \dots, f_{n-d}) \rightarrow \text{Spec } R[T_1, \dots, T_n] = \mathbb{A}_R^n$.)

Proposition 4.42. *Let $f: X \rightarrow S$ be smooth of relative dimension d at $x \in X$. Then there exists an open neighborhood U of x such that the restriction $\Omega_{X/Y|U} (= \Omega_{U/Y})$ is free of rank d .*

Theorem 4.43. *Let k be an algebraically closed field, and let X be an irreducible k -scheme of finite type. Let $d = \dim X$. Then X is smooth over k if and only if $\Omega_{X/k}$ is locally free of rank d .*

Proposition 4.44. *Let $f: X \rightarrow S$ be smooth of relative dimension d at $x \in X$. Then there exists an open neighborhood U of x such that the restriction $U \rightarrow S$ of f to U is formally smooth.*

Theorem 4.45. *Let $f: X \rightarrow Y$ be a morphism locally of finite presentation (e.g., if Y is noetherian and f is locally of finite type). Then f is smooth if and only if f is formally smooth.*

We skip the proof that smoothness implies formal smoothness, see for instance [Bo] Ch. 8.5. (But cf. the previous proposition which shows that a smooth morphism is at least “locally formally smooth”.)

5. PROJECTIVE SCHEMES *

(5.1) The Proj construction.

Reference: [GW1] Ch. 13.

Definition 5.1.

- (1) A graded ring is a ring A with a decomposition $A = \bigoplus_{d \geq 0} A_d$ as abelian groups such that $A_d \cdot A_e \subseteq A_{d+e}$ for all d, e . The elements of A_d are called homogeneous of degree d .
- (2) Let R be a ring. A graded R -algebra is a graded ring A together with a ring homomorphism $R \rightarrow A$.
- (3) A homomorphism $A \rightarrow B$ of graded rings (or graded R -algebras) is a ring homomorphism (or R -algebra homomorphism, respectively) $f: A \rightarrow B$ such that $f(A_d) \subseteq B_d$ for all d .
- (4) Let A be a graded ring. A graded A -module is an A -module M with a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ such that $A_d \cdot M_e \subseteq M_{d+e}$ for all d, e . The elements of M_d are called homogeneous of degree d .
- (5) A homomorphism $M \rightarrow N$ of graded A -modules is an A -module homomorphism $f: M \rightarrow N$ such that $f(M_d) \subseteq N_d$ for all d .
- (6) Let A be a graded ring and let M be a graded A -module. A homogeneous submodule of M is a submodule $N \subseteq M$ such that $N = \bigoplus_{d \in \mathbb{Z}} (N \cap M_d)$. In this way, N is itself a graded A -module and the inclusion $N \hookrightarrow M$ is a homomorphism of graded A -modules. (And conversely, every injective homomorphism of graded A -modules has a homogeneous submodule as its image.) A homogeneous submodule of A is called a homogeneous ideal.

Example 5.2. Let R be a ring. Then the polynomial ring $R[T_0, \dots, T_n]$ is a graded R -algebra if we set $R[T_0, \dots, T_n]_d$ to be the R -submodule of homogeneous polynomials of degree d .

We now fix a graded ring A .

For a homogeneous element $f \in A_e$, and a graded A -module M , the localization M_f is a graded A -module via

$$M_{f,d} = \left\{ \frac{m}{f^i}; m \in M_{d+ei} \right\}.$$

Applying this to A as an A -module, we obtain a grading on A_f giving A_f the structure of a graded ring. Then M_f is a graded A_f -module.

We define

$$M_{(f)} := M_{f,0},$$

the degree 0 part of M_f . Then $A_{(f)}$ is a ring and $M_{(f)}$ is an $A_{(f)}$ -module.

Example 5.3. Let R be a ring. Then

$$R[T_0, \dots, T_n]_{(T_i)} = R\left[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right].$$

Definition 5.4. We write $A_+ := \bigoplus_{d>0} A_d$, an ideal of A . A homogeneous prime ideal $\mathfrak{p} \subset A$ is called *relevant* if $A_+ \not\subseteq \mathfrak{p}$.

Definition 5.5. We denote by $\text{Proj}(A)$ the set of all relevant homogeneous prime ideals of A . We equip $\text{Proj}(A)$ with the Zariski topology, by saying that the closed subsets are the subsets of the form

$$V_+(I) := \{\mathfrak{p} \in \text{Proj}(A); I \subseteq \mathfrak{p}\}.$$

for homogeneous ideals $I \subseteq A$.

For a homogeneous element f , we write $D_+(f) := \text{Proj}(A) \setminus V_+(f)$.

Lemma 5.6. Let $f \in A$ be a homogeneous element. Then the map

$$D_+(f) \rightarrow \text{Spec } A_{(f)}, \quad \mathfrak{p} \mapsto (\mathfrak{p}A_f) \cap A_{(f)}$$

is a homeomorphism.

Proposition 5.7. There is a unique sheaf $\mathcal{O}$ of rings on $\text{Proj}(A)$ such that

$$\Gamma(D_+(f), \mathcal{O}) = A_{(f)}$$

for every homogeneous element $f \in A$ and with restriction maps given by the canonical maps between the localizations. The ringed space $(\text{Proj}(A), \mathcal{O})$ is a separated scheme which we again denote by $\text{Proj}(A)$.

Definition 5.8. Let R be a ring, and let X be an R -scheme. We say that X is *projective over R* (or that the morphism $X \rightarrow \text{Spec } R$ is projective), if there exist $n \geq 0$ and a closed immersion $X \rightarrow \mathbb{P}_R^n$ of R -schemes.

Theorem 5.9. Let R be a ring, and let X be a projective R -scheme. Then X is proper over R .

(5.2) Quasi-coherent modules on $\text{Proj}(A)$.

Let A be a graded ring, $X = \text{Proj } A$. If M is a graded A -module, there is a unique sheaf $\widetilde{M}$ of $\mathcal{O}_X$ -modules such that

$$\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$$

for every homogeneous element $f \in A$, and such that the restriction maps for inclusions of the form $D_+(g) \subseteq D_+(f)$ are given by the natural maps between the localizations. This sheaf is a quasi-coherent $\mathcal{O}_X$ -module.

Example 5.10. Let $A(n)$ be the graded A -module defined by $A(n) = \bigoplus_{d \in \mathbb{Z}} A_{n+d}$. We set $\mathcal{O}_X(n) = \widetilde{A(n)}$. If $A = R[T_0, \dots, T_n]$ for a ring R , so that $X = \mathbb{P}_R^n$, then this notation is consistent with our previous definition.

For $f \in A_d$, multiplication by f^k defines an isomorphism

$$\mathcal{O}_{X|D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(n)|_{D_+(f)}.$$

In particular, if A is generated as an A_0 -algebra by A_1 , then $\mathcal{O}_X(n)$ is a line bundle.

Assume, for the remainder of this section, that A is generated as an A_0 -algebra by A_1 . So $X = \bigcup_{f \in A_1} D_+(f)$, and $\mathcal{O}_X(n)$ is a line bundle.

For an $\mathcal{O}_X$ -module $\mathcal{F}$, write $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$, and define a graded A -module $\Gamma_*(\mathcal{F})$ by

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)).$$

Lemma 5.11. *For a graded A -module M , there is a natural map $M \rightarrow \Gamma_*(\widetilde{M})$. For an $\mathcal{O}_X$ -module $\mathcal{F}$, there is a natural map $\widetilde{\Gamma_*(\mathcal{F})} \rightarrow \mathcal{F}$. If $\mathcal{F}$ is quasi-coherent, then the latter map is an isomorphism.*

Call a graded A -module M *saturated*, if the map $M \rightarrow \Gamma_*(\widetilde{M})$ is an isomorphism.

Proposition 5.12. *The functors $M \rightarrow \widetilde{M}$ and $\mathcal{F} \rightarrow \Gamma_*(\mathcal{F})$ define an equivalence of categories between the category of saturated graded A -modules and the category of quasi-coherent $\mathcal{O}_X$ -modules.*

6. COHOMOLOGY OF $\mathcal{O}_X$ -MODULES

General references: [We], [HS], [Gr], [KS], [GW2].

The formalism of derived functors.

We will start by developing the formalism of derived functors and use it to attach to any scheme (and even to any ringed space) X and sheaf $\mathcal{F}$ of abelian groups its *cohomology groups* $H^i(X, \mathcal{F})$ which “measure the failure of exactness of the global section functor” in the sense that $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ and for every short exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

of sheaves of abelian groups on X we have a *long exact cohomology sequence*

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{G}) \rightarrow H^0(X, \mathcal{H}) \rightarrow \\ H^1(X, \mathcal{F}) \rightarrow H^1(X, \mathcal{G}) \rightarrow H^1(X, \mathcal{H}) \rightarrow \\ H^2(X, \mathcal{F}) \rightarrow \dots \end{aligned}$$

In a similar way, we can attach “right derived functors” to any left exact functor between abelian categories (with the source satisfying the mild additional condition of “having enough injectives”, see below).

One of the applications (with quite a bit of further work) of this formalism is a proof of the Riemann-Roch theorem (cf. Problem 33 for a preliminary version).

(6.1) Complexes in abelian categories.

Reference: [We] Ch. 1, [GW2] App. F.

Let $\mathcal{A}$ be an abelian category (e.g., the category of abelian groups, the category of R -modules for a ring R , the category of abelian sheaves on a topological space X , the category of $\mathcal{O}_X$ -modules on a ringed space X , or the category of quasi-coherent $\mathcal{O}_X$ -modules on a scheme X ; see [GW2] Section (F.7)).

A *complex* in $\mathcal{A}$ is a sequence of morphisms

$$\dots \longrightarrow A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \longrightarrow \dots$$

in $\mathcal{A}$ ($i \in \mathbb{Z}$), such that $d^{i+1} \circ d^i = 0$ for every $i \in \mathbb{Z}$. The maps d^i are called the *differentials* of the complex.

Given complexes $A^\bullet, B^\bullet$, a morphism $A^\bullet \rightarrow B^\bullet$ of complexes is a family of morphisms $f^i: A^i \rightarrow B^i$ such that the f^i commute with the differentials of $A^\bullet$ and $B^\bullet$ in the obvious way. With this notion of morphisms, we obtain the category $C(\mathcal{A})$ of complexes in $\mathcal{A}$. This is an abelian category (kernels, images, ... are formed degree-wise); see [We] Thm. 1.2.3.

Definition 6.1. Let $A^\bullet$ be a complex in $\mathcal{A}$. For $i \in \mathbb{Z}$, we call

$$H^i(A^\bullet) := \ker(d^i) / \operatorname{im}(d^{i-1})$$

the i -th cohomology object of $A^\bullet$. We obtain functors $H^i: C(\mathcal{A}) \rightarrow \mathcal{A}$. We say that $A^\bullet$ is exact at i , if $H^i(A^\bullet) = 0$. We say that $A^\bullet$ is exact, if $H^i(A^\bullet) = 0$ for all i .

Remark 6.2. Let $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ be a sequence of morphisms of complexes. The sequence is exact (in the sense that at each point the kernel and image in the category $C(\mathcal{A})$ coincide) if and only if for every i , the sequence $0 \rightarrow A^i \rightarrow B^i \rightarrow C^i \rightarrow 0$ is exact.

Proposition 6.3. [[We] Thm. 1.3.1] Let $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ be an exact sequence of complexes in $\mathcal{A}$. Then there are maps $\delta^i: H^i(C^\bullet) \rightarrow H^{i+1}(A^\bullet)$ (called boundary maps) such that, together with the maps induced by functoriality of the H^i , we obtain the long exact cohomology sequence

$$\cdots H^i(A^\bullet) \rightarrow H^i(B^\bullet) \rightarrow H^i(C^\bullet) \rightarrow H^{i+1}(A^\bullet) \rightarrow \cdots$$

Remark 6.4.

- (1) Most of the proofs in this section are omitted. Usually, they are straightforward and it is more enlightening (and does not take more time) to do them yourself than to read them in one of the references.
- (2) When proofs are given, we usually assume that the underlying categories are the category of abelian groups or of R -modules over some ring, so that we can argue with elements of the objects of our category in order to construct maps and check equality of maps. It is usually a mechanical procedure to translate such proofs into proofs that apply to arbitrary abelian categories. An alternative for small categories is to use Mitchell's embedding theorem which states that any small abelian category admits a fully faithful exact additive functor to the category of R -modules (for some not necessarily commutative ring R). This means that constructions such as kernels, cokernels and images can be carried out equivalently in the original category or with the corresponding R -modules, and for the latter "arguments involving elements" can be used.

We need a criterion which ensures that two morphisms between complexes induce the same maps on all cohomology objects. See [We] 1.4.

Definition 6.5. Let $f, g: A^\bullet \rightarrow B^\bullet$ be morphisms of complexes. We say that f and g are homotopic, if there exists a family of maps $k^i: A^i \rightarrow B^{i-1}$, $i \in \mathbb{Z}$, such that

$$f - g = dk + kd,$$

which we use as short-hand notation for saying that for every i ,

$$f^i - g^i = d_B^{i-1} \circ k^i + k^{i+1} \circ d_A^i.$$

In this case we write $f \sim g$. The family $(k^i)_i$ is called a homotopy.

Proposition 6.6. *Let $f, g: A^\bullet \rightarrow B^\bullet$ be morphisms of complexes which are homotopic. Then for every i , the maps $H^i(A^\bullet) \rightarrow H^i(B^\bullet)$ induced by f and g are equal.*

In particular, if $A^\bullet$ is a complex such that $\text{id}_{A^\bullet} \sim 0$, then $H^i(A^\bullet) = 0$ for all i , i.e., $A^\bullet$ is exact.

Definition 6.7. *Let $A^\bullet$ and $B^\bullet$ be complexes. We say that $A^\bullet$ and $B^\bullet$ are homotopy equivalent, if there exist morphisms $f: A^\bullet \rightarrow B^\bullet$ and $g: B^\bullet \rightarrow A^\bullet$ of complexes such that $g \circ f \sim \text{id}_A$ and $f \circ g \sim \text{id}_B$. In this case, f and g induce isomorphisms $H^i(A^\bullet) \cong H^i(B^\bullet)$ for all i .*

(6.2) Left exact functors.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. All functors $F: \mathcal{A} \rightarrow \mathcal{B}$ that we consider are assumed to be additive, i.e., they induce group homomorphisms $\text{Hom}_{\mathcal{A}}(A, A') \rightarrow \text{Hom}_{\mathcal{B}}(F(A), F(A'))$ for all A, A' in $\mathcal{A}$.

Definition 6.8. *A (covariant) functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is called left exact, if for every short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ in $\mathcal{A}$, the sequence*

$$0 \rightarrow F(A') \rightarrow F(A) \rightarrow F(A'')$$

is exact.

Definition 6.9. *A contravariant functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is called left exact if for every short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ in $\mathcal{A}$, the sequence*

$$0 \rightarrow F(A'') \rightarrow F(A) \rightarrow F(A')$$

is exact.

Similarly, we have the notion of right exact functor. A functor which is left exact and right exact (and hence preserves exactness of arbitrary sequences) is called exact.

Let $A_0 \in \mathcal{A}$. Then the functors $A \mapsto \text{Hom}_{\mathcal{A}}(A, A_0)$ and $A \mapsto \text{Hom}_{\mathcal{A}}(A_0, A)$ are left exact.

(6.3) δ -functors.

Reference: [We] 2.1, [GW2] Section (F.48).

Let $\mathcal{A}, \mathcal{B}$ be abelian categories.

Definition 6.10. *A δ -functor from $\mathcal{A}$ to $\mathcal{B}$ is a family $(T^i)_{i \geq 0}$ of functors $\mathcal{A} \rightarrow \mathcal{B}$ together with morphisms $\delta^i: T^i(A'') \rightarrow T^{i+1}(A')$ (called boundary morphisms) for every short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ in $\mathcal{A}$, such that the sequence*

$$0 \rightarrow T^0(A') \rightarrow T^0(A) \rightarrow T^0(A'') \rightarrow T^1(A') \rightarrow \dots$$

is exact, and such that the δ^i are compatible with morphisms of short exact sequences in the obvious way.

Definition 6.11. A δ -functor $(T^i)_i$ from $\mathcal{A}$ to $\mathcal{B}$ is called universal, if for every δ -functor $(S^i)_i$ and every morphism $f^0: T^0 \rightarrow S^0$ of functors, there exist unique morphisms $f^i: T^i \rightarrow S^i$ of functors for all $i > 0$, such that the f^i , $i \geq 0$ are compatible with the boundary maps δ^i of the two δ -functors for each short exact sequence in $\mathcal{A}$.

The definition implies that given a (left exact) functor F , any two universal δ -functors $(T^i)_i$, $(T'^i)_i$ with $T^0 = T'^0 = F$ are isomorphic (in the obvious sense) via a unique isomorphism.

Definition 6.12. A functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is called effaceable, if for every X in $\mathcal{A}$ there exists a monomorphism $\iota: X \hookrightarrow A$ with $F(\iota) = 0$.

A particular case is the situation where each X admits a monomorphism to an object I with $F(I) = 0$.

Proposition 6.13. ([We] Thm. 2.4.7, Ex. 2.4.5.) Let $(T^i)_i$ be a δ -functor from $\mathcal{A}$ to $\mathcal{B}$ such that for every $i > 0$, the functor T^i is effaceable. Then $(T^i)_i$ is a universal δ -functor.

(6.4) Injective objects.

Let $\mathcal{A}$ be an abelian category.

Definition 6.14. An object I in $\mathcal{A}$ is called injective, if the functor $X \mapsto \text{Hom}_{\mathcal{A}}(X, I)$ is exact.

If I is injective, then every short exact sequence $0 \rightarrow I \rightarrow A \rightarrow A'' \rightarrow 0$ in $\mathcal{A}$ splits. Conversely, if I is an object such that every short exact sequence $0 \rightarrow I \rightarrow A \rightarrow A'' \rightarrow 0$ splits, then I is injective. Since exactness of split sequences is preserved by any additive functor (the sequence obtained by applying the functor is again split exact), it is not surprising that injective objects play a special role in the construction of right derived functors.

Definition 6.15. Let $X \in \mathcal{A}$. An injective resolution of X is an exact sequence

$$0 \rightarrow X \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$$

in $\mathcal{A}$, where every I^i is injective.

Definition 6.16. We say that the category $\mathcal{A}$ has enough injectives if for every object X there exists a monomorphism $X \hookrightarrow I$ from X into an injective object I . Equivalently: Every object has an injective resolution.

Remark 6.17.

- (1) The categories of abelian groups, of R -modules for a ring R , of abelian sheaves on a topological space, and more generally of $\mathcal{O}_X$ -modules on a

ringed space X all have enough injective objects. For these categories, it is not too hard to show this statement directly (for $\mathcal{O}_X$ -modules on a ringed space X , e.g., see [H] Proposition III.2.2). Alternatively, one can show that they all are “Grothendieck abelian categories”, and that those have enough injective objects (see [GW2] Sections (F.12), (21.2)).

- (2) Dually, we have the notion of *projective* object (i.e., P such that $\text{Hom}_{\mathcal{A}}(P, -)$ is exact), of *projective resolution* $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$, and of abelian categories with *enough projective objects*. For a ring R , the category of R -modules clearly has enough projectives, since every free module is projective, and every module admits an epimorphism from a free module. Categories of sheaves of abelian groups or $\mathcal{O}_X$ -modules typically do not have enough projectives.

Lemma 6.18. ([We] Theorem 2.3.7, see also Porism 2.2.7) *Let $\mathcal{A}$ be an abelian category. Let $0 \rightarrow A \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots$ be exact with $A, A^i \in \mathcal{A}$. Let $0 \rightarrow B \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ be a complex with $B, I^i \in \mathcal{A}$ and all I^i injective. For every morphism $f: A \rightarrow B$ in $\mathcal{A}$ there exists a morphism $g: A^\bullet \rightarrow I^\bullet$ of complexes such that the diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & A^0 & \longrightarrow & A^1 \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & I^0 & \longrightarrow & I^1 \longrightarrow \cdots \end{array}$$

is commutative, and g is uniquely determined up to homotopy.

Lemma 6.19. (cf. [We] Lemma 2.2.8, “Horseshoe lemma”) *Let $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ be an exact sequence in an abelian category $\mathcal{A}$. Let $0 \rightarrow A_1 \rightarrow I_1^\bullet$ and $0 \rightarrow A_3 \rightarrow I_3^\bullet$ be injective resolutions. Let $I_2^i := I_1^i \times I_3^i$ and define $A_2 \rightarrow I_2^0$ by using the composition $A_2 \rightarrow A_3 \rightarrow I_3^0$ and lifting the map $A_1 \rightarrow I_1^0$ to a map $A_2 \rightarrow I_2^0$, using the injectivity of I_2^0 . Then we obtain an injective resolution $0 \rightarrow A_2 \rightarrow I_2^\bullet$, and a term-wise split short exact sequence*

$$0 \longrightarrow I_1^\bullet \longrightarrow I_2^\bullet \longrightarrow I_3^\bullet \longrightarrow 0$$

of complexes.

(6.5) Right derived functors.

Theorem 6.20. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor, and assume that $\mathcal{A}$ has enough injectives.*

For each $A \in \mathcal{A}$, fix an injective resolution $0 \rightarrow A \rightarrow I^\bullet$, and define

$$R^i F(A) = H^i(F(I^\bullet)), \quad i \geq 0,$$

where $F(I^\bullet)$ denotes the complex obtained by applying the functor to all I^i and to the differentials of the complex $I^\bullet$. Then:

- (1) The object $R^i F A$ is independent of the choice of injective resolution of A up to natural isomorphism. There is a natural way to extend the above definition to morphisms, and the $R^i F$ are additive functors $\mathcal{A} \rightarrow \mathcal{B}$.
- (2) We have an isomorphism $F \cong R^0 F$ of functors.
- (3) For I injective, we have $R^i F I = 0$ for all $i > 0$.
- (4) The family $(R^i F)_i$ admits natural boundary maps making it a universal δ -functor.

We call the $R^i F$ the right derived functors of F .

Remarks on the proof. For part (1), the main ingredient is Lemma 6.18. Note that whenever morphisms of complexes are homotopic, this remains true after applying any functor, since we may also apply the functor to the homotopy morphisms. Parts (2) and (3) are easy. To construct the boundary maps giving rise to long exact cohomology sequences in Part (4), use Lemma 6.19; since the exact sequence $0 \rightarrow I_1^\bullet \rightarrow I_2^\bullet \rightarrow I_3^\bullet \rightarrow 0$ is term-wise split, the exactness is preserved when applying the functor F . Then we may apply Proposition 6.3. Finally, all $R^i F$, $i > 0$ are effaceable by (c) and hence this δ -functor is universal. $\square$

Definition 6.21. Let F be a left exact functor as above. We say that an object $A \in \mathcal{A}$ is F -acyclic, if $R^i F(A) = 0$ for all $i > 0$.

Definition 6.22. Let F be a left exact functor as above, and let $A \in \mathcal{A}$. An F -acyclic resolution of A is an exact sequence $0 \rightarrow A \rightarrow J^0 \rightarrow J^1 \rightarrow \dots$ where all J^i are F -acyclic.

Proposition 6.23. Let F be a left exact functor as above, and let $A \in \mathcal{A}$. Let $0 \rightarrow A \rightarrow J^0 \rightarrow J^1 \rightarrow \dots$ be an F -acyclic resolution. Then we have natural isomorphisms $R^i F(A) = H^i(F(J^\bullet))$, i.e., we can compute $R^i F(A)$ by an F -acyclic resolution.

Sketch of the proof. The assertion is easy to see for $i = 0$. For $i = 1$, note that $\text{Ker}(F(J^1) \rightarrow F(J^2)) = F(J^0/A)$ by the left exactness of F . From this, one gets $R^1 F(A) = H^1(F(J^\bullet))$.

From the long exact cohomology sequence attached to the short exact sequence $0 \rightarrow A \rightarrow J^0 \rightarrow J^0/A \rightarrow 0$, we get isomorphisms $R^i F(J^0/A) \cong R^{i+1} F(A)$ for $i \geq 1$. Since $0 \rightarrow J^0/A \rightarrow J^1 \rightarrow \dots$ is an acyclic resolution of J^0/A , we get

$$R^{i+1} F(A) \cong R^i F(J^0/A) \cong H^{i+1}(F(J^\bullet)),$$

where the second isomorphism holds by induction. $\square$

(6.6) Derived categories.

Sometimes it is useful to employ the language of derived categories which gives rise to a notion of derived functor RF that contains slightly more

information than the family $(R^i F)_i$ discussed above (for a left exact functor $F: \mathcal{A} \rightarrow \mathcal{B}$). In fact, RF attaches to each object of $\mathcal{A}$, and more generally to each complex A in $C(\mathcal{A})$, a *complex* $RF(A)$ (up to some equivalence relation; more precisely, an object $RF(A) \in D(\mathcal{B})$ in the derived category of $\mathcal{B}$) whose cohomology objects are the $R^i F(A)$, i.e., $H^i(RF(A)) = R^i F(A)$.

See [GW2] (and the references given there) for a systematic treatment: Appendix F for the general theory, in particular Sections (F.37) for the definition of the derived category of an abelian category, (F.42), (F.43), (F.44) for the notion of derived functor, (F.48) for a comparison with the notion of δ -functor defined above, Chapter 21 for general results on cohomology of $\mathcal{O}_X$ -modules on a ringed space X (specifically, the derived functors of the global section and direct image functors), Chapter 22 for cohomology of quasi-coherent $\mathcal{O}_X$ -modules on a scheme X , Chapter 23 for cohomology of proper schemes and the derived direct image functor of a proper morphism. While in these chapters one sometimes obtains stronger or clearer statements using derived categories, for a large part the key arguments are exactly the same as with the “naive approach” using derived functors that we will take. In Chapter 24 on the theorem on formal functions and in particular in Chapter 25 on Grothendieck duality, the machinery of derived categories really shows its full power (but it is unlikely that we will get there in this class).

Cohomology of sheaves.

General reference: [H] Ch. III, [Stacks], [GW2] Chapter 21.

(6.7) Cohomology groups.

Let X be a topological space. Denote by (Ab_X) the category of abelian sheaves (i.e., sheaves of abelian groups) on X . We have the global section functor

$$\Gamma: (\text{Ab}_X) \rightarrow (\text{Ab}), \quad \mathcal{F} \mapsto \Gamma(X, \mathcal{F}),$$

to the category of abelian groups. This is a left exact functor, and we denote its right derived functors by $H^i(X, -)$. We call $H^i(X, \mathcal{F})$ the *i-th cohomology group of X with coefficients in $\mathcal{F}$* .

Example 6.24. For a field k , $H^1(\mathbb{P}_k^1, \mathcal{O}(-2)) \neq 0$.

(6.8) Flasque sheaves.

Reference: [GW2] Section (21.7).

Definition 6.25. Let X be a topological space. A sheaf $\mathcal{F}$ on X is called *flasque* (or *flabby*), if all restriction maps $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for $V \subseteq U \subseteq X$ open are surjective.

For the next lemma, we use the *extension by zero* functor: Let $j: U \rightarrow X$ be the inclusion of an open subspace into a topological space X (similarly one can work with an open immersion of ringed spaces). For an abelian sheaf $\mathcal{F}$ on U we define $j_!\mathcal{F}$ as the sheafification of the presheaf

$$V \mapsto \begin{cases} \mathcal{F}(V) & \text{if } V \subseteq U, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain a left adjoint to the restriction functor j^* .

Lemma 6.26. Let X be a ringed space. Let $\mathcal{F}$ be an injective object in the category of $\mathcal{O}_X$ -modules. Then $\mathcal{F}$ is flasque.

Sketch of proof. Identify $\mathcal{F}(U) = \text{Hom}(j_{U!}\mathcal{O}_U, \mathcal{F})$, where $j_U: U \rightarrow X$ is the inclusion of an open subset U into X , and $j_{U!}$ is the extension by zero functor, using the adjunction between $j_{U!}$ and j_U^* . $\square$

Proposition 6.27. Let X be a topological space, and let $\mathcal{F}$ be a flasque abelian sheaf on X . Then $\mathcal{F}$ is Γ -acyclic, i.e., $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Proof. Given a flasque sheaf $\mathcal{F}$; embed it into an injective abelian sheaf and use the results of Problem 34 and dimension shift. $\square$

Corollary 6.28. Let X be a ringed space. The right derived functors of the global section functor from the category of $\mathcal{O}_X$ -modules to the category of abelian groups can naturally be identified with $H^i(X, -)$.

It follows that for an $\mathcal{O}_X$ -module $\mathcal{F}$ the cohomology groups $H^i(X, \mathcal{F})$ carry a natural $\Gamma(X, \mathcal{O}_X)$ -module structure.

(6.9) Grothendieck vanishing.

Reference: [H] III.2, [GW2] Section (21.12).

Lemma 6.29. Let X be a topological space, and let $\iota: Y \rightarrow X$ be the inclusion of a closed subset Y of X . Let $\mathcal{F}$ be an abelian sheaf on X . Then there are natural isomorphisms

$$H^i(Y, \mathcal{F}) = H^i(X, \iota_*\mathcal{F}), \quad i \geq 0.$$

Sketch of proof. Use that ι_* preserves the property of being flasque, that the cohomology groups can be computed using a flasque resolution and that ι_* is exact (here we use that ι is a closed embedding). $\square$

Proposition 6.30. Let X be a noetherian topological space, and let $(\mathcal{F}_i)_{i \in I}$ be a filtered inductive system of abelian sheaves on X . Then the natural

homomorphism

$$\operatorname{colim}_i H^n(X, \mathcal{F}_i) \rightarrow H^n(X, \operatorname{colim}_i \mathcal{F})$$

is an isomorphism for all $n \geq 0$.

Sketch of proof. First show the statement for global sections, i.e., for $n = 0$ (this says precisely that on a noetherian topological space the colimit presheaf is actually a sheaf; we do not have to sheafify). Since filtered colimits are exact, both sides give δ -functors from the abelian category of inductive systems of abelian sheaves (with fixed index set I) to the category of abelian groups. To conclude the proof, it is enough to show that they are both effaceable (because this implies they are universal, and universal δ -functors are determined by their 0-th term).

To show effaceability, take a system $(\mathcal{F}_i)_i$ and consider a functorial embedding into a flasque sheaf of the $\mathcal{F}_i$ (such as the Godement construction, i.e., $\mathcal{G}_i(U) = \prod_{x \in U} \mathcal{F}_{i,x}$). Since filtered colimits are exact and, on noetherian spaces, preserve flasqueness, this gives rise to a flasque resolution of $\operatorname{colim}_i \mathcal{F}_i$. With this at hand, it is not hard to finish the proof. $\square$

Let X be a topological space, $U \subseteq X$ open and $Z = X \setminus U$ its closed complement. Denote by $j: U \rightarrow X$ and $i: Z \rightarrow X$ the inclusions. Recall the *extension by zero* functor $j_!: (\operatorname{Ab}_U) \rightarrow (\operatorname{Ab}_X)$ (which is left adjoint to the pull-back j^{-1}). From this adjunction and the one between i^* and i_* we obtain the maps in the short exact sequence

$$0 \rightarrow j_!(\mathcal{F}|_U) \rightarrow \mathcal{F} \rightarrow i_*(\mathcal{F}|_Z) \rightarrow 0$$

of abelian sheaves on X . The exactness can be checked on stalks where it is clear.

Theorem 6.31. (Grothendieck) *Let X be a noetherian topological space (i.e., the descending chain condition holds for closed subsets of X), let $n = \dim X$, and let $\mathcal{F}$ be a sheaf of abelian groups on X . Then*

$$H^i(X, \mathcal{F}) = 0 \quad \text{for all } i > n.$$

Proof. The proof is omitted in these notes. It consists of a series of reduction steps using the above ingredients, ultimately reducing to the fact that the cohomology of the constant sheaf $\mathbb{Z}_X$ on an irreducible space vanishes in positive degrees (since there a constant sheaf is flasque). See the references given above for more. $\square$

Čech cohomology.

Reference: [GW2] Sections (21.14) to (21.17), [H] III.4, [Stacks] 01ED (and following sections); a classical reference is [Go].

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(6.10) Čech cohomology groups.

Let X be a topological space, and let $\mathcal{F}$ be a presheaf of abelian groups on X .

Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of X . We fix a total order of the index set I (but see below for a sketch that the results are independent of this). For $i_0, \dots, i_p \in I$, we write $U_{i_0 \dots i_p} := \bigcap_{\nu=0}^p U_{i_\nu}$.

We define

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \Gamma(U_{i_0 \dots i_p}, \mathcal{F})$$

and

$$d: C^p(\mathcal{U}, \mathcal{F}) \rightarrow C^{p+1}(\mathcal{U}, \mathcal{F}), \quad (s_{\underline{i}})_{\underline{i}} \mapsto \left(\sum_{\nu=0}^{p+1} (-1)^\nu s_{i_0 \dots \widehat{i}_\nu \dots i_p | U_{\underline{i}} |_{\underline{i}}} \right)_{\underline{i}},$$

where $\widehat{}$ indicates that the corresponding index is omitted. One checks that $d \circ d = 0$, so we obtain a complex, the so-called *Čech complex for the cover $\mathcal{U}$ with coefficients in $\mathcal{F}$* .

Definition 6.32. *The Čech cohomology groups for $\mathcal{U}$ with coefficients in $\mathcal{F}$ are defined as*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(C^\bullet(\mathcal{U}, \mathcal{F})), \quad p \geq 0.$$

Since $\mathcal{F}$ is a sheaf, we have $\check{H}^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$. In fact, a presheaf $\mathcal{F}$ is a sheaf if and only if for all open subsets $U \subseteq X$ and all covers $\mathcal{U}$ of U the natural map $\Gamma(U, \mathcal{F}) \rightarrow \check{H}^0(\mathcal{U}, \mathcal{F})$ is an isomorphism.

(6.11) The “full” Čech complex.

Instead of the “ordered” Čech complex as above, we can also consider the “full” Čech complex

$$C_f^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0, \dots, i_p} \Gamma(U_{i_0 \dots i_p}, \mathcal{F}),$$

with differentials defined by the same formula as above. Then the projection $C_f^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow C^\bullet(\mathcal{U}, \mathcal{F})$ is a homotopy equivalence, with “homotopy inverse” given by

$$(s_{\underline{i}})_{\underline{i}} \mapsto (t_{\underline{i}})_{\underline{i}},$$

where $t_{\underline{i}} = 0$ whenever two entries in the multi-index $\underline{i}$ coincide, and otherwise $t_{\underline{i}} = \text{sgn}(\sigma) s_{\sigma(\underline{i})}$, where σ is the permutation such that $\sigma(\underline{i})$ is in increasing order.

In particular, we have natural isomorphisms between the cohomology groups of the two complexes. So we also see that the Čech cohomology groups as defined above are independent of the choice of order on I .

(6.12) Passing to refinements.

Definition 6.33. A refinement of a cover $\mathcal{U} = (U_i)_i$ of X is a cover $\mathcal{V} = (V_j)_{j \in J}$ (with J totally ordered) together with a map $\lambda: J \rightarrow I$ respecting the orders on I and J such that $V_j \subseteq U_{\lambda(j)}$ for every $j \in J$.

Given a refinement $\mathcal{V}$ of $\mathcal{U}$, one obtains a natural map (using restriction of sections to smaller open subsets)

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^p(\mathcal{V}, \mathcal{F}).$$

We can pass to the colimit over all these maps given by refinements, and define

$$\check{H}^p(X, \mathcal{F}) := \operatorname{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F}),$$

the p -th Čech cohomology group of X with coefficients in $\mathcal{F}$. More precisely, we here take the colimit over the category of refinements (see [GW2] Section (F.3)). Working with the full Čech complex, one can view this as a colimit over a partially ordered set “as usual”, for the cohomology groups and even for the Čech complexes, if one considers them in the homotopy category (i.e., the map between complexes attached to a refinement is independent of λ up to homotopy), cf. [GW2] Section (21.16), also for the discussion of set-theoretic issues.

Proposition 6.34. Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be a short exact sequence of abelian sheaves on X . Then there exists a natural homomorphism $\delta: \Gamma(X, \mathcal{F}'') \rightarrow \check{H}^1(X, \mathcal{F}')$ such that the sequence

$$\begin{aligned} 0 \rightarrow \Gamma(X, \mathcal{F}') \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}'') \\ \rightarrow \check{H}^1(X, \mathcal{F}') \rightarrow \check{H}^1(X, \mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{F}'') \end{aligned}$$

is exact. (But note that the sequence does not continue after $\check{H}^1(X, \mathcal{F}'')$.)

Proof. This can be checked “directly”. For instance, to construct the connecting homomorphism δ , take an element $s'' \in \Gamma(X, \mathcal{F}'')$. Locally on X , we can lift it to sections of $\mathcal{F}$, so we find an open cover $\mathcal{U} = (U_i)_i$ and an element $(s_i)_i \in C^0(\mathcal{U}, \mathcal{F})$ mapping to $(s''|_{U_i})_i \in C^0(\mathcal{U}, \mathcal{F}'')$.

Its image $(s_j - s_i)_{i < j}$ in $C^1(\mathcal{U}, \mathcal{F})$ will usually be different from 0 (in fact it is = 0 if and only if s is in the image of $\Gamma(X, \mathcal{F})$), but has image 0 in $C^1(\mathcal{U}, \mathcal{F}'')$, and hence comes from an element $(t_{ij})_{i < j} \in C^1(\mathcal{U}, \mathcal{F}')$. Then $(t_{ij})_{i < j}$ has image 0 in $C^2(\mathcal{U}, \mathcal{F}')$ (because that is obviously true in $C^2(\mathcal{U}, \mathcal{F})$ and the morphism $\mathcal{F}' \rightarrow \mathcal{F}$ is injective), and so gives rise to a class in $\check{H}^1(\mathcal{U}, \mathcal{F}')$. Its image in $\check{H}^1(X, \mathcal{F}')$ is the image of s under δ . One checks that this procedure is independent of choices and gives rise to the exact sequence in the proposition. $\square$

(6.13) Comparison of cohomology and Čech cohomology.

In degrees 0 and 1, cohomology and Čech cohomology coincide. For degree 0, we have already shown this, so we proceed to the case of degree 1. We start with some preparations.

We define a sheaf version of the Čech complex as follows:

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{\underline{i}=(i_0 < \dots < i_p)} j_{\underline{i},*}(\mathcal{F}|_{U_{\underline{i}}}),$$

with differentials defined by (basically) the same formula as above. Here $j_{\underline{i}}$ denotes the inclusion $U_{\underline{i}} \hookrightarrow X$.

We have a natural map $\mathcal{F} \rightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F})$, which on an open V is given by $s \mapsto (s|_{U_i \cap V})_i$.

Proposition 6.35. *The sequence $0 \rightarrow \mathcal{F} \rightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \rightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \rightarrow \dots$ is exact.*

Proof. The exactness can be checked on stalks, and one can show that for each point of X , the stalks of the above complex form a complex that is homotopy equivalent to 0. We omit the details. $\square$

Proposition 6.36. *If $\mathcal{F}$ is flasque, then all $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ are flasque, and $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. It is not hard to check that the sheaves $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ are flasque, since all the constructions involved preserve flasqueness.

We then obtain

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(\Gamma(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}))) = H^p(X, \mathcal{F}) = 0,$$

where the second equality holds since $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a flasque resolution of $\mathcal{F}$ by the above, and the third one follows since $\mathcal{F}$ itself is flasque. $\square$

Proposition 6.37. *Let X be a topological space and let $\mathcal{F}$ be an abelian sheaf on X .*

- (1) *Let $\mathcal{U}$ be an open cover of X . For every i , there is a natural map $\check{H}^i(\mathcal{U}, \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$.*
- (2) *These maps are compatible with refinements, so we obtain a natural map $\check{H}^i(X, \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$. These maps are functorial in $\mathcal{F}$.*
- (3) *For $i = 0, 1$, the natural map $\check{H}^i(X, \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$ is an isomorphism.*

Proof. Part (1) follows from Proposition 6.18 applied to the resolution $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ and any injective resolution. We omit the proof of Part (2).

For Part (3), it remains to consider the case $i = 1$. Embed $\mathcal{F}$ into a flasque sheaf $\mathcal{G}$. We obtain short exact sequences

$$\Gamma(X, \mathcal{G}) \rightarrow \Gamma(X, \mathcal{G}/\mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{F}) \rightarrow 0$$

by Proposition 6.34 Proposition 6.36 and

$$\Gamma(X, \mathcal{G}) \rightarrow \Gamma(X, \mathcal{G}/\mathcal{F}) \rightarrow H^1(X, \mathcal{F}) \rightarrow 0$$

since flasque sheaves are $\Gamma(X, -)$ -acyclic. The statement follows from this. $\square$

One can also show that the natural map $\check{H}^2(X, \mathcal{F}) \rightarrow H^2(X, \mathcal{F})$ is always injective.

The following result will allow us to compute cohomology groups of separated schemes with coefficients in quasi-coherent modules as Čech cohomology.

Theorem 6.38. (Cartan's Theorem) *Let X be a ringed space, and let $\mathcal{B}$ be a basis of the topology of X which is stable under finite intersections. Let $\mathcal{F}$ be an $\mathcal{O}_X$ -module. Assume that $\check{H}^i(U, \mathcal{F}) = 0$ for all $U \in \mathcal{B}$ and $i > 0$. Then*

- (1) *we have $H^i(U, \mathcal{F}) = 0$ for all $U \in \mathcal{B}$ and $i > 0$,*
- (2) *The natural homomorphisms $\check{H}^i(\mathcal{U}, \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$ are isomorphisms (of $\Gamma(X, \mathcal{O}_X)$ -modules, functorially in $\mathcal{F}$) for all $i \geq 0$ and all covers $\mathcal{U}$ of X consisting of elements of $\mathcal{B}$.*
- (3) *The natural homomorphisms $\check{H}^i(X, \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$ are isomorphisms for all $i \geq 0$.*
- (4) *Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be a short exact sequence of $\mathcal{O}_X$ -modules, all satisfying the above assumption, and let $\mathcal{U}$ be an open cover of X by elements of $\mathcal{B}$. Then the sequence*

$$0 \rightarrow C^\bullet(\mathcal{U}, \mathcal{F}') \rightarrow C^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow C^\bullet(\mathcal{U}, \mathcal{F}'') \rightarrow 0$$

induced by the short exact sequence is exact, and the induced long exact sequence (Proposition 6.3) is identified via the isomorphisms in Part (2) (similarly, passing to the colimit over all open covers, for Part (3)) with the long exact cohomology sequence for ordinary cohomology, i.e., in both ways one obtains the same boundary morphisms.

See e.g., [Go] II Thm. 5.9.2, [Stacks] 01E0 or [GW2] Section (21.17). For Part (4), see [Go] II Section 5.11, cf. also Remark 6.45.

Remark* 6.39. From the point of view of derived categories the key point is the following. Fix an open cover $\mathcal{U}$ of X . (A similar description holds after passing to the colimit over covers of X .) We want to write the derived functor $R\Gamma$ of the global section functor as a composition of functors. To get started, we write Γ itself as the composition

$$(\mathcal{O}_X\text{-Mod}) \xrightarrow{\iota} (\text{P-}\mathcal{O}_X\text{-Mod}) \xrightarrow{\check{H}^0(\mathcal{U}, -)} (\Gamma(X, \mathcal{O}_X)\text{-Mod})$$

where the category in the middle is the category of *presheaves* of $\mathcal{O}_X$ -modules. The functor ι is the natural inclusion. (We could also write Γ as the composition of ι and the global section functor on the category of presheaves. However, since the latter is exact(!), it is its own derived functor and therefore this would not yield any interesting information about $R\Gamma$ and in particular would not allow to make a connection with Čech cohomology.)

Both the functors ι and $\check{H}^0$ above are left exact and we obtain (using that $R\iota$ maps injective sheaves to $\check{H}^0(\mathcal{U}, -)$ -acyclic presheaves)

$$R\Gamma = R\check{H}^0(\mathcal{U}, -) \circ R\iota.$$

To derive “explicit” consequences from this equality one computes that the right derived functors are $R^i\iota(\mathcal{F}) = \mathcal{H}^i(\mathcal{F})$, where $\mathcal{H}^i(\mathcal{F})$ is the presheaf $U \mapsto H^i(U, \mathcal{F})$, and $(R^i\check{H}^0(\mathcal{U}, -))(\mathcal{F}) = \check{H}^i(\mathcal{U}, \mathcal{F})$. Then one can use the theory of spectral sequences. In particular, the *Grothendieck spectral sequence* (see [Gr], [GW2] Section (F.50)) yields a precise connection between $R^i\Gamma(\mathcal{F}) = H^i(X, \mathcal{F})$ and the $(R^i\check{H}^0(\mathcal{U}, -))(R^j\iota\mathcal{F}) = \check{H}^i(\mathcal{U}, \mathcal{H}^j(\mathcal{F}))$. Unsurprisingly, the situation simplifies considerably whenever many of the latter terms vanish, as for instance in the situation of Cartan’s theorem.

Cohomology of affine schemes.

General references: [H] Ch. III, [Stacks], [GW2] Chapter 22.

(6.14) Vanishing of cohomology of quasi-coherent sheaves on affine schemes.

Theorem 6.40. *Let $X = \operatorname{Spec}(R)$ be an affine scheme, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ -module. Then $\check{H}^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. It is enough to show that $\check{H}^i(\mathcal{U}, \mathcal{F}) = 0$ for all finite covers $\mathcal{U}$ of X by principal open subsets. So say $X = \bigcup_{i=1}^r D(f_i)$ and let $M = \Gamma(X, \mathcal{F})$, so $\mathcal{F} = M^\sim$. We then need to show that the complex (the “augmented (full) Čech complex” where we have inserted M on the left)

$$0 \rightarrow M \rightarrow \prod_{i_0} M_{f_{i_0}} \rightarrow \prod_{i_0, i_1} M_{f_{i_0}f_{i_1}} \rightarrow \cdots$$

is exact.

This can be proved using “direct computation” (see [GW1] Lemma 12.33). Alternatively, we may rewrite the sequence as

$$0 \rightarrow M \rightarrow M \otimes_R R' \rightarrow M \otimes_R R' \otimes_R R' \rightarrow \cdots,$$

where we set $R' = \prod_{i=1}^r R_{f_i}$, so that $\operatorname{Spec}(R') = \bigsqcup D(f_i)$ is the disjoint union of the open subschemes $D(f_i)$. Then the ring homomorphism $R \rightarrow R'$ is faithfully flat, and the exactness is a consequence of the theory of “faithfully flat descent” (specifically Problem 39, see also [GW1] Lemma 14.64). $\square$

Using Cartan’s Theorem, Theorem 6.38, we get:

Theorem 6.41. *Let X be an affine scheme, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ -module. Then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Theorem 6.42. *Let X be a separated scheme, and let $\mathcal{U}$ be a cover of X by affine open subschemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ -module. Then the natural homomorphisms $\check{H}^i(\mathcal{U}, \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$ are isomorphisms for all $i \geq 0$.*

Recall that X is separated, if the morphism $X \rightarrow \operatorname{Spec} \mathbb{Z}$ is separated. This is true whenever X is separated over any affine scheme (or over any separated scheme). In this case, the intersection of any two affine open subschemes of X is again affine. This is what we need for Cartan's theorem, and the above theorem also holds under this (slightly) weaker assumption.

Remark 6.43. For X noetherian, the use of Cartan's Theorem can be avoided by using the result (see [H] III.3) that for a noetherian ring A , $X = \operatorname{Spec} A$, and I an injective A -module, the $\mathcal{O}_X$ -module $\tilde{I}$ is a flasque $\mathcal{O}_X$ -module.

Together with the fact that for an affine scheme X the global section functor is exact on the category of quasi-coherent $\mathcal{O}_X$ -modules, this gives the vanishing of $H^i(X, \mathcal{F})$ for X affine, $\mathcal{F}$ quasi-coherent and $i > 0$.

It also implies that any quasi-coherent $\mathcal{O}_X$ -module on a noetherian scheme can be embedded into a flasque *quasi-coherent* sheaf. From this one can prove Theorem 6.42.

See [H] III.3, Theorem III.4.5; cf. also [GW2] Section (22.18).

Corollary 6.44. *Let X be a separated scheme which can be covered by $n + 1$ affine open subschemes. Then $H^i(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$ -module $\mathcal{F}$ and every $i > n$.*

Proof. For $\mathcal{U}$ a cover by $n + 1$ affine open subschemes and $p > n$ we have $C^p(\mathcal{U}, \mathcal{F}) = 0$, so $H(X, \mathcal{F}) = \check{H}^p(\mathcal{U}, \mathcal{F}) = 0$. $\square$

Remark 6.45. Let X be a quasi-compact separated scheme. It is not difficult to show that every quasi-coherent $\mathcal{O}_X$ -module can be embedded into a quasi-coherent $\mathcal{O}_X$ -module whose cohomology in degrees > 0 vanishes ([Stacks] OBDY). Therefore the functors $\mathcal{F} \mapsto H^p(X, \mathcal{F})$ are effaceable for all $p > 0$ and thus the family $(H^p(X, -))_p$ of functors *restricted to the abelian category of quasi-coherent $\mathcal{O}_X$ -modules* is a universal δ -functor. Since Čech cohomology (for a fixed open cover $\mathcal{U}$ by affine open subschemes) on the category of quasi-coherent $\mathcal{O}_X$ -modules is a δ -functor, we obtain homomorphisms $H^p(X, \mathcal{F}) \rightarrow \check{H}^p(\mathcal{U}, \mathcal{F})$. These are the inverses of the homomorphisms constructed above. This gives another proof of Part (4) of Theorem 6.38 in the setting at hand.

The following lemma and theorem had to be omitted in the class.

Lemma 6.46. *Let X be a scheme. For $f \in \Gamma(X, \mathcal{O}_X)$ write*

$$X_f = \{x \in X; f(x) \neq 0 \in \kappa(x)\},$$

an open subset of X which we consider as an open subscheme.

If there exist $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that X_{f_i} is affine for $i = 1, \dots, n$ and such that $f_1, \dots, f_n$ generate the unit ideal in the ring $\Gamma(X, \mathcal{O}_X)$, then X is affine.

Theorem 6.47. (Serre's criterion for affineness) *Let X be a quasi-compact scheme. The following are equivalent:*

- (i) *The scheme X is affine.*
- (ii) *For every quasi-coherent $\mathcal{O}_X$ -module $\mathcal{F}$ and every $i > 0$, $H^i(X, \mathcal{F}) = 0$.*
- (iii) *For every quasi-coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, $H^1(X, \mathcal{I}) = 0$.*

Proof. See [H] Theorem III.3.7 or [GW1] Theorem 12.35. □

Cohomology of projective schemes.

(6.15) The cohomology of line bundles on projective space.

References: [H] III.5, [GW2] Section (22.6), [Stacks] 01XS.

Using Čech cohomology, we can compute the cohomology of line bundles on projective space. It is best to aggregate the results for all $\mathcal{O}(d)$, as we have already seen for their global sections, a result which we repeat as the first statement below.

Theorem 6.48. *Let A be a ring, $n \geq 1$, $S = A[T_0, \dots, T_n]$, $X = \mathbb{P}_A^n$. Then*

- (1) *There is a natural isomorphism $S \cong \bigoplus_{d \in \mathbb{Z}} H^0(X, \mathcal{O}(d))$.*
- (2) *For $i \neq 0, n$ and all $d \in \mathbb{Z}$ we have $H^i(X, \mathcal{O}(d)) = 0$.*
- (3) *There is a natural isomorphism $H^n(X, \mathcal{O}(-n-1)) \cong A$.*
- (4) *For every d , there is a perfect pairing*

$$H^0(X, \mathcal{O}(d)) \times H^n(X, \mathcal{O}(-d-n-1)) \rightarrow H^n(X, \mathcal{O}(-n-1)) \cong A,$$

i.e., a bilinear map which induces isomorphisms

$$H^0(X, \mathcal{O}(d)) \cong H^n(X, \mathcal{O}(-d-n-1))^\vee$$

and

$$H^0(X, \mathcal{O}(d))^\vee \cong H^n(X, \mathcal{O}(-d-n-1))$$

(where $-^\vee = \text{Hom}_A(-, A)$ denotes the A -module dual).

Sketch of proof. We compute the cohomology groups as Čech cohomology groups for the standard cover $\mathcal{U} = (D_+(T_i))_i$ of $\mathbb{P}_A^n$. It simplifies the reasoning to do the computation for all $\mathcal{O}(d)$ at once, i.e., to compute the cohomology groups of $\mathcal{F} := \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d)$ (and to – implicitly – keep track of the grading by d). Since cohomology is compatible with direct sums (cf. Proposition 6.30; we proved that for noetherian schemes, but it holds more generally for quasi-compact separated schemes, cf. [GW2] Corollary 21.56), this also gives the result for the individual $\mathcal{O}(d)$.

The Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ is

$$0 \rightarrow \prod_i S_{T_i} \rightarrow \prod_{i < j} S_{T_i T_j} \rightarrow \cdots \rightarrow S_{T_0 \dots T_n} \rightarrow 0$$

(with non-zero entries in degrees $0, \dots, n$), cf. the computation of the global sections of the sheaves $\mathcal{O}(d)$, Proposition 3.51. Also note that that proposition proves Part (1) of the theorem here. From this, we see that we can identify

$$H^n(\mathbb{P}_A^n, \mathcal{F}) = \bigoplus_{i_0, \dots, i_n < 0} A \cdot T_0^{i_0} \cdots T_n^{i_n} \subseteq S[T_0^{-1}, \dots, T_n^{-1}].$$

The degree d part on the left corresponds to the part of the direct sum where $\sum i_\nu = d$. In particular it vanishes whenever $d \geq -n$ and is of rank 1 for $d = -n - 1$ (since then all $i_\nu = -1$). This easily implies Parts (3) and (4). In (4), the term $H^n(X, \mathcal{O}(-d - n - 1))$ can be identified with the free A -module generated by the monomials $T_0^{i_0} \cdots T_n^{i_n}$ such that $\sum i_\nu = -d - n - 1$ and all $i_\nu < 0$, and the pairing is given by

$$\langle T_0^{i_0} \cdots T_n^{i_n}, T_0^{j_0} \cdots T_n^{j_n} \rangle = \begin{cases} 1 & \text{if } i_\nu + j_\nu = -1 \ \forall \nu \\ 0 & \text{otherwise.} \end{cases}$$

It remains to prove the vanishing statement of Part (2) for $0 < i < n$. We do this by induction on n . Note that in view of Part (1), all the cohomology groups $H^i(\mathbb{P}_A^n, \mathcal{F})$ carry a natural S -module structure.

First note that for the localization we have $H^i(\mathbb{P}_A^n, \mathcal{F})_{T_n} = 0$. In fact, this localization is the i -th cohomology group of the localized Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})_{T_n}$ which computes the cohomology $H^\bullet(D_+(T_n), \mathcal{F}|_{D_+(T_n)})$ which vanishes in positive degrees. Therefore it suffices to show that multiplication by T_n is a bijection $H^i(\mathbb{P}_A^n, \mathcal{F}) \rightarrow H^i(\mathbb{P}_A^n, \mathcal{F})$ for all $0 < i < n$.

The global section $T_n \in H^0(\mathbb{P}_A^n, \mathcal{O}(1))$ gives rise to a short exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}_{\mathbb{P}_A^n} \rightarrow \mathcal{O}_{V_+(T_n)} \rightarrow 0,$$

and tensoring with the locally free module $\mathcal{F}$ we obtain a short exact sequence

$$0 \rightarrow \mathcal{F} \otimes \mathcal{O}(-1) \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \rightarrow 0$$

where $\mathcal{F}' = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}_A^n}} \mathcal{O}_{V_+(T_n)}$ is (the push-forward from $V_+(T_n) \cong \mathbb{P}_A^{n-1}$ to $\mathbb{P}_A^n$) of the sheaf analogous to $\mathcal{F}$ on $\mathbb{P}_A^{n-1}$.

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Multiplication by $T_n \in \Gamma(\mathbb{P}_A^n, \mathcal{O}(1))$ gives an isomorphism $\mathcal{F} \otimes \mathcal{O}(-1) \rightarrow \mathcal{F}$. Therefore (using Lemma 6.29) the long exact cohomology sequence attached to the above short exact sequence can be written as

$$\cdots \rightarrow H^i(\mathbb{P}_A^n, \mathcal{F}) \rightarrow H^i(\mathbb{P}_A^n, \mathcal{F}) \rightarrow H^i(\mathbb{P}_A^{n-1}, \mathcal{F}') \rightarrow \cdots$$

where the maps $H^i(\mathbb{P}_A^n, \mathcal{F}) \rightarrow H^i(\mathbb{P}_A^n, \mathcal{F})$ are given by multiplication by T_n (i.e., what we want to show is that these maps are isomorphisms for $0 < i < n$). The induction hypothesis and the observations that the map $H^0(\mathbb{P}_A^n, \mathcal{F}) \rightarrow H^0(\mathbb{P}_A^{n-1}, \mathcal{F}')$ is surjective (cf. Part (1)) and that the map $H^{n-1}(\mathbb{P}_A^{n-1}, \mathcal{F}') \rightarrow H^n(\mathbb{P}_A^n, \mathcal{F})$ is injective, then allow us to conclude.

For the injectivity cf. the above proof for Part (3). Looking at individual graded pieces of $H^{n-1}(\mathbb{P}_A^{n-1}, \mathcal{F}')$ and the kernel of $H^n(\mathbb{P}_A^n, \mathcal{F}) \rightarrow H^n(\mathbb{P}_A^n, \mathcal{F})$, we have a surjective homomorphism of free A -modules of the same rank which is necessarily an isomorphism.

In fact, the kernel of $H^n(\mathbb{P}_A^n, \mathcal{F}) \rightarrow H^n(\mathbb{P}_A^n, \mathcal{F})$ is the free A -module spanned by all monomials of the form $T_0^{i_0} \cdots T_{n-1}^{i_{n-1}} T_n^{-1}$ with all $i_\nu < 0$. The homomorphism $H^{n-1}(\mathbb{P}_A^{n-1}, \mathcal{F}') \rightarrow H^n(\mathbb{P}_A^n, \mathcal{F})$ is given by mapping $T_0^{i_0} \cdots T_{n-1}^{i_{n-1}}$ to $T_0^{i_0} \cdots T_{n-1}^{i_{n-1}} T_n^{-1}$. To verify this, one can use that the long exact cohomology sequence for the short exact sequence $0 \rightarrow \mathcal{F} \otimes \mathcal{O}(-1) \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \rightarrow 0$ can be computed in terms of Čech cohomology, cf. Theorem 6.38 (4) and Remark 6.45. $\square$

(6.16) Finiteness of cohomology of coherent $\mathcal{O}_X$ -modules on projective schemes.

References: [H] III.5; [GW2] Sections (23.1), (23.2).

Definition 6.49. *Let X be a noetherian scheme. An $\mathcal{O}_X$ -module $\mathcal{F}$ is called coherent, if it is quasi-coherent and of finite type.*

Let A be a noetherian ring. For an $\mathcal{O}_{\mathbb{P}_A^n}$ -module $\mathcal{F}$, we write $\mathcal{F}(d) := \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}_A^n}} \mathcal{O}(d)$. We need the following general lemma.

Lemma 6.50. *Let X be a quasi-compact and separated scheme, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$ -module, and let $s \in \Gamma(X, \mathcal{L})$ be a global section. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ -module.*

- (1) *Let $t \in \Gamma(X, \mathcal{F})$ be a global section such that $t|_{X_s} = 0$. Then there exists an integer $n > 0$ such that $t \otimes s^{\otimes n} = 0 \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$.*
- (2) *For every section $t' \in \Gamma(X_s, \mathcal{F})$ there exist $n > 0$ and a section $t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ such that $t|_{X_s} = t' \otimes s^{\otimes n}$.*

Proof. If X is affine and $\mathcal{L} = \mathcal{O}_X$, then this follows immediately from our results on quasi-coherent $\mathcal{O}_X$ -modules; namely we know that then $\Gamma(X_s, \mathcal{F}) = \Gamma(X, \mathcal{F})_s$. For the general case, let $X = \bigcup_i U_i$ be a finite affine open cover such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$ for all i (and fix such isomorphisms).

Then (1) can be checked on each U_i individually and thus follows from what was said in the beginning. To prove Part (2), we construct t by considering sections t_i on the U_i obtained from the restrictions $t|_{U_i \cap X_s}$, using the result in the affine case. The t_i may not glue, but applying Part (1) to the intersections $U_i \cap U_j$ and the elements $t_i|_{U_i \cap U_j} - t_j|_{U_i \cap U_j}$ we find that for n sufficiently large, the $t_i \otimes s^{\otimes n}$ will glue to a section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$. See [GW1] Theorem 7.22 or [H] Lemma II.5.14 for more details. $\square$

The condition that X be separated cannot be dropped in the previous lemma, but can be replaced by a weaker assumption. Do you see where it was used and which weaker condition on X would suffice?

Proposition 6.51. *Let A be a noetherian ring, $n \geq 1$, let $X = \mathbb{P}_A^n$, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ -module.*

(1) *There exist integers $d_1, \dots, d_s$ and a surjective $\mathcal{O}_X$ -module homomorphism*

$$\bigoplus_{i=1}^s \mathcal{O}(d_i) \rightarrow \mathcal{F}.$$

(2) *For d sufficiently large, the $\mathcal{O}_{\mathbb{P}_A^n}$ -module $\mathcal{F}(d)$ is globally generated, i.e., there exist $N \geq 0$ and a surjective homomorphism $\mathcal{O}_{\mathbb{P}_A^n}^N \rightarrow \mathcal{F}(d)$.*

Sketch of proof. It is easy to see that (1) and (2) are equivalent. (To go from (1) to (2), use that for every $d \geq 0$ there is a surjection $\mathcal{O}_X^N \rightarrow \mathcal{O}(d)$.) We prove (2). For $i \in \{0, \dots, n\}$, $\mathcal{F}|_{D_+(T_i)}$ has the form $\widetilde{M}_i$ for some finitely generated $A[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}]$ -module M_i . For any $s \in M_i = \Gamma(D_+(T_i), \mathcal{F})$, for d sufficiently large, $X_i^d s$ extends to a global section of $\mathcal{F}(d)$, by Lemma 6.50. Combining these extensions for generating systems for each of the M_i gives the desired surjection. $\square$

Theorem 6.52. *Let A be a noetherian ring, X be a projective A -scheme, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ -module. Then for all $i \geq 0$, the A -module $H^i(X, \mathcal{F})$ is finitely generated.*

Sketch of proof. Using Lemma 6.29, we reduce to the case that $X = \mathbb{P}_A^n$. (It is easy to check that the property of being coherent is preserved under push-forward along a closed immersion.) By Proposition 6.51 we find a short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0,$$

where $\mathcal{E} = \bigoplus_{i=1}^s \mathcal{O}(d_i)$ is already known to have the desired property by our explicit computation of the cohomology of $\mathcal{O}(d_i)$. The kernel $\mathcal{K}$ is again coherent because X is noetherian. Consider the corresponding long exact cohomology sequence

$$\cdots \rightarrow H^i(X, \mathcal{K}) \rightarrow H^i(X, \mathcal{E}) \rightarrow H^i(X, \mathcal{F}) \rightarrow H^{i+1}(X, \mathcal{K}) \rightarrow \cdots.$$

We now argue by descending induction on i , the induction start being give by the vanishing result Corollary 6.44. Thus we may assume that $H^{i+1}(X, \mathcal{G})$ is

finitely generated for every coherent $\mathcal{O}_X$ -module $\mathcal{G}$; we apply this to $\mathcal{K}$. As was already mentioned, $H^i(X, \mathcal{E})$ is also known to be finitely generated. Since A is noetherian, the exactness implies that $H^i(X, \mathcal{F})$ is finitely generated (for every coherent $\mathcal{O}_X$ -module $\mathcal{F}$). $\square$

At this point it is not hard to prove that higher derived images $R^i f_* \mathcal{F}$ of a coherent $\mathcal{O}_X$ -module under a projective morphism $f: X \rightarrow Y$ are coherent (see [H] III.8).

Another useful result is the following vanishing statement.

Proposition 6.53. *Let A be a noetherian ring, $\iota: X \rightarrow \mathbb{P}_A^n$ a closed immersion, $\mathcal{L} = \iota^* \mathcal{O}_{\mathbb{P}_A^n}(1)$, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ -module. For $n \in \mathbb{Z}$ we write $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$. Then there exists $n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$ and all $i > 0$, we have $H^i(X, \mathcal{F}(n)) = 0$.*

Sketch of proof. Using the projection formula (Part (1) of the proposition below), one reduces to the case $X = \mathbb{P}_A^n$. It is then clear from the above, that the statement holds whenever $\mathcal{F}$ is a direct sum of line bundles $\mathcal{O}(d)$. The general statement follows from this by descending induction, similarly as above. $\square$

Proposition 6.54. (Projection formula) *Let $f: X \rightarrow Y$ be a morphism of schemes, let $\mathcal{F}$ be an $\mathcal{O}_X$ -module, and let $\mathcal{G}$ be an $\mathcal{O}_Y$ -module. There is a natural homomorphism*

$$(f_* \mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G} \rightarrow f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G})$$

which is an isomorphism if one of the following conditions is satisfied:

- (1) $\mathcal{G}$ is a locally free $\mathcal{O}_Y$ -module,
- (2) f is a closed immersion,
- (3) $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent, and for every affine open $V \subseteq Y$, $f^{-1}(V)$ is affine,
- (4) $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent, f is quasi-compact and separated, and $\mathcal{G}$ is a flat $\mathcal{O}_Y$ -module.

Sketch of proof. The homomorphism is obtained formally using the adjunction between f^* and f_* . The statement that it is an isomorphism is local on Y . Thus for (1), we may assume that $\mathcal{G}$ is free, and then the claim is easy to check. In situation (2) one can check that the homomorphism induces an isomorphism on each stalk, and hence is an isomorphism. Under the assumptions in (3), one may assume that Y and X are affine and the claim then follows easily from the description of pushforward and pullback in this case (Proposition 2.17). For further details, and for an argument to prove the statement in case (4), see [GW2] Proposition 22.80. $\square$

Serre duality.**(6.17) The Theorem of Riemann–Roch revisited.**

References: [H] III.7, IV.1; [GW2] Chapters 25, 26.

Recall the Theorem of Riemann–Roch that we stated above (Theorem 3.45). In this section, we prove a preliminary version, which also gives a more conceptual view on the “error term” $\dim \Gamma(X, \mathcal{O}(K - D))$ (with notation as above).

Let k be an algebraically closed field.

Definition 6.55. *Let X be a projective k -scheme, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ -module. We call*

$$\chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F})$$

the Euler characteristic of $\mathcal{F}$.

Note that the sum is finite (by the Grothendieck vanishing theorem, Theorem 6.31) and that each term is finite by the results of the previous section.

Now let X/k be a smooth, projective, connected curve. Then $\chi(\mathcal{F}) = \dim_k H^0(X, \mathcal{F}) - \dim_k H^1(X, \mathcal{F})$.

The following theorem is the preliminary version of the Theorem of Riemann–Roch mentioned above.

Theorem 6.56. *Let $\mathcal{L}$ be a line bundle on X . Then*

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) + \chi(\mathcal{O}_X).$$

Sketch of proof. See also Problem 33. The statement is clear for $\mathcal{L} = \mathcal{O}_X$. Since every line bundle is isomorphic to the line bundle attached to a Weil divisor, it is therefore enough to prove that for every closed point $x \in X$, and every line bundle $\mathcal{L}$ on X , we have

$$\chi(\mathcal{L}) = \chi(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-[x])) + 1$$

We write $\mathcal{L}(D) := \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ for any divisor D . The short exact sequence

$$0 \rightarrow \mathcal{O}_X(-[x]) \rightarrow \mathcal{O}_X \rightarrow \kappa(x) \rightarrow 0$$

remains exact after tensoring with $\mathcal{L}$, so we obtain a short exact sequence

$$0 \rightarrow \mathcal{L}(-[x]) \rightarrow \mathcal{L} \rightarrow \kappa(x) \rightarrow 0.$$

Since the Euler characteristic is additive in short exact sequences (use the long exact cohomology sequence) and since $\chi(\kappa(x)) = 1$, the claim follows. $\square$

Now we can define the genus of X as $g := 1 - \chi(\mathcal{O}_X) = \dim_k H^1(X, \mathcal{O}_X)$. From the above, we immediately get

Corollary 6.57. (Theorem of Riemann) *Let $\mathcal{L}$ be a line bundle on X . Then*

$$\dim_k H^0(X, \mathcal{L}) \geq \deg(\mathcal{L}) + 1 - g.$$

Furthermore, the Theorem of Riemann–Roch, Theorem 3.45, follows from the above result on Euler characteristics and the Serre duality theorem (which however we cannot prove in this class).

The following material was only sketched in class, and you do not have to know it for the exam (unless you want to talk about it).

Remark 6.58. In the statement of the theorem below the property of a scheme of being *Cohen-Macaulay* is mentioned. A scheme X is called *Cohen-Macaulay*, if each of the local rings $\mathcal{O}_{X,x}$ is Cohen-Macaulay in the sense of commutative algebra. Also recall that a scheme X is called *regular*, if each local ring $\mathcal{O}_{X,x}$ is regular, i.e., $\dim \mathcal{O}_{X,x} = \dim_{\kappa(x)} \mathfrak{m}_x / \mathfrak{m}_x^2$ (the latter quantity, by the Lemma of Nakayama, is equal to the minimal number of elements needed to generate the maximal ideal $\mathfrak{m}$). See Definition 4.6 for the definition of a *smooth morphism*. As usual, we say that a k -scheme X is smooth (or that X is smooth over k), if the structure morphism $X \rightarrow \operatorname{Spec}(k)$ is smooth. Consider the following properties.

- (i) X is smooth over k ,
- (ii) X is regular,
- (iii) X is regular in codimension 1 (i.e., for each x such that $\dim(\mathcal{O}_{X,x}) = 1$, the ring $\mathcal{O}_{X,x}$ is regular (and hence a discrete valuation ring)) and Cohen-Macaulay,
- (iv) X is Cohen-Macaulay,
- (v) X is normal.

We then have the following implications.

$$\begin{array}{ccccccc} \text{(i)} & \implies & \text{(ii)} & \implies & \text{(iii)} & \implies & \text{(iv)} \\ & & & & \Downarrow & & \\ & & & & \text{(v)} & & \end{array}$$

If k is perfect (e.g., if k is algebraically closed), then also $(\text{ii}) \Rightarrow (\text{i})$ is true.

Theorem 6.59. (Serre duality) *Let k be an algebraically closed field and let X be a connected proper k -scheme which is Cohen-Macaulay. There is a unique (up to isomorphism) coherent $\mathcal{O}_X$ -module ω , the so-called dualizing sheaf, such that for every locally free $\mathcal{O}_X$ -module $\mathcal{E}$ of finite rank on X , there is a natural isomorphism*

$$H^{n-i}(X, \mathcal{E}) \cong H^i(X, \mathcal{E}^{-1} \otimes \omega)^\vee$$

of k -vector spaces (where $-^\vee$ denotes the dual k -vector space).

If X is smooth over k , then the dualizing sheaf is a line bundle and coincides with the so-called *canonical bundle*, the top exterior power of the sheaf $\Omega_{X/k}^1$ of differentials of X over k , see Chapter 4.

A fairly elementary approach in the case where X is projective is to prove a similar duality theorem for the cohomology of line bundles on projective space $\mathbb{P}_k^n$ (this we have basically done above, Theorem 6.48 (4), with $\omega = \mathcal{O}(-n-1)$), and then to derive the statement for (certain) closed subschemes of projective space; see [H] III.7. A more general (but technically more sophisticated) approach is to derive this duality result from the existence of a right adjoint functor $f^\times$ of the derived push-forward functor $Rf_*: D_{\text{qcoh}}(X) \rightarrow D_{\text{qcoh}}(\text{Spec } k)$, where $f: X \rightarrow \text{Spec } k$ is the structure morphism and $D_{\text{qcoh}}(X)$ denotes the full triangulated subcategory of the derived category $D(X)$ of the category of $\mathcal{O}_X$ -modules (and this generalizes to the case of arbitrary proper morphisms f between noetherian schemes). In fact, one shows that for X Cohen-Macaulay and equidimensional of dimension n the complex $f^\times k$ is concentrated in degree $-n$, and we denote by ω the unique non-vanishing cohomology object. For every quasi-coherent $\mathcal{O}_X$ -module $\mathcal{F}$ and integer i one then has

$$\begin{aligned} H^i(X, \mathcal{F})^\vee &= \text{Hom}_k(Rf_* \mathcal{F}[i], k) = \text{Hom}_{D(X)}(\mathcal{F}[i], \omega[n]) \\ &= \text{Ext}_{\mathcal{O}_X}^{n-i}(\mathcal{F}, \omega), \end{aligned}$$

and if $\mathcal{F}$ is locally free of finite rank, then we can identify the right hand side with

$$\text{Ext}_{\mathcal{O}_X}^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega) = H^{n-i}(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega).$$

This approach also gives a result (in terms of the derived category) without requiring that X be Cohen-Macaulay, and also for (proper) morphisms $f: X \rightarrow Y$ where Y is not necessarily the spectrum of a field. See [GW2] Chapter 25 for more on this and for further references.

As another application of Serre duality, we mention the following result. The first part is named the Lemma of Enriques-Severi-Zariski. To name an application, we note that this is one of the ingredients of the proof that every regular proper surface over a field is projective (see [GW2] Theorem 25.151).

The result still holds (but in Part (1) only for H^1 , as stated, while for X Cohen-Macaulay one has an analogous result for all H^i , $i < n$) if the assumption that X be Cohen-Macaulay is replaced by the assumption that X is normal. On the other hand it is clear that the assumption that X has dimension at least 2 cannot be dropped.

Theorem 6.60. *Let k be an (algebraically closed) field, let X be an integral Cohen-Macaulay (e.g., smooth) projective k -scheme and let $\iota: X \rightarrow \mathbb{P}_k^N$ be a closed immersion of k -schemes.*

Assume that $\dim X \geq 2$.

- (1) Fix $d > 0$ and let $\mathcal{L} := \iota^* \mathcal{O}_{\mathbb{P}_k^N}(d)$. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$ -module of finite rank. We write $\mathcal{E}(n) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$, $n \in \mathbb{Z}$. Then for $n \ll 0$, $H^1(X, \mathcal{E}(n)) = 0$.
- (2) Let $f \in k[T_0, \dots, T_N]$ be a non-constant homogeneous polynomial, and let $Z := V_+(f)$, a closed subscheme of $\mathbb{P}_k^N$. Then $X \cap Z$ is connected.

Sketch of proof. Part (1) follows from Serre duality (Theorem 6.59) and the vanishing result of Proposition 6.53. To prove Part (2), let d denote the degree of f . We view Z as an effective Cartier divisor. Then $\mathcal{O}_{\mathbb{P}_k^N}(Z) \cong \mathcal{O}(d)$. We set $\mathcal{L} = \iota^* \mathcal{O}(d)$ and $\mathcal{E} = \mathcal{O}_X$ and apply Part (1) to find n such that $H^1(X, \mathcal{L}^{-n}) = 0$. Let Z_n be the Cartier divisor $n \cdot Z$ (with associated line bundle $\cong \mathcal{O}(dn)$). We then have a short exact sequence

$$0 \rightarrow \mathcal{L}^{-n} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X \cap Z_n} \rightarrow 0,$$

where $X \cap Z_n$ denotes the scheme-theoretic intersection of X and Z , i.e., $\mathcal{O}_{X \cap Z_n} = \iota^* \mathcal{O}_{Z_n}$. The underlying topological space of $X \cap Z_n$ is independent of $n \geq 1$ and equals the set-theoretic intersection $X \cap Z$. It is therefore sufficient to show that the scheme $X \cap Z_n$ is connected. But the above short exact sequence induces, in view of the vanishing of $H^1(X, \mathcal{L}^{-n})$ given by Part (1) and since $H^0(X, \mathcal{L}^{-n}) = 0$ (cf. Remark 3.62), an isomorphism $\Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(X, \mathcal{O}_{X \cap Z_n})$ of k -vector spaces. Since $\Gamma(X, \mathcal{O}_X) = k$ (cf. Problem 22), it follows that $\Gamma(X, \mathcal{O}_{X \cap Z_n}) = k$, as well, and in particular $X \cap Z_n$ is connected. See also [GW2] Section (25.28), [H] Section III.7. $\square$

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