ALGEBRAIC GEOMETRY 2, SUMMER TERM 2026. LECTURE COURSE NOTES.

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Introduction

This lecture course is a continuation of the course *Algebraic Geometry 1* which covered the definition of schemes, and some basic notions about schemes and scheme morphisms, such as immersions and subschemes, fiber products of schemes and separated morphisms.

The main object of study of this term's course will be the notion of $\mathscr{O}_X$-module, a natural analogue of the notion of module over a ring in the context of sheaves of rings. As we will see, the $\mathscr{O}_X$-modules on a scheme X contain a lot of information about the geometry of this scheme, and we will study them using a variety of methods. In particular, we will introduce the notion of *cohomology groups*, a powerful algebraic tool that makes its appearance in many areas of algebra and geometry.

These notes are based on the notes for similar courses which I taught in 2023 and 2019, and will be updated or modified to reflect the content of the current class while the term proceeds.

References. The books [GW1], [GW2] by Wedhorn and myself, by Hartshorne [H], and by Mumford [Mu]. More precise references are given in most of the individual sections. Mumford still uses the ancient terminology and calls a *prescheme* what we call a scheme, and a *scheme* what we call a separated scheme. Further references: [Stacks], [EGA].

We give a brief overview of some of the topics we will discuss in this class.

Proper schemes and morphisms. (Section (1.6)) Similarly as separatedness is the algebro-geometric version of the Hausdorff property, properness is the algebro-geometric version of compactness; we will see that - as expected, cf. the Riemann sphere - projective space and all of its closed subschemes are proper. (The converse is, interestingly, not true: There exist proper schemes (over a field, say) which cannot be embedded into any projective space.)) April 14, 2026

Divisors and line bundles. (Chapter 3) Assume we have a closed subscheme $X \subseteq \mathbb{P}^n_k$. We can intersect X with the "coordinate hyperplanes" $V_+(X_i)$ to obtain subschemes of X of a very special form, so-called divisors.

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We will discuss several perspectives on the notion of divisor. It is an interesting question whether, given a divisor, it arises in the way described above.

$\mathscr{O}_X$-Modules. (Chapter 2) Similarly to the notion of module over a ring, there is a notion of sheaves of modules over a sheaf of rings which we can apply in particular to the structure sheaf of a scheme or of any ringed space. Such $\mathscr{O}_X$-modules contain a lot of interesting information about the underlying space X. There is a close connection to the notion of divisor which we will discuss in the course. (And for this reason, in the course we will first talk about $\mathscr{O}_X$-modules, and then about divisors.)

Cohomology of sheaves. (Chapter 6) We know that a surjective sheaf morphism might not induce surjections on the sets of sections over a fixed open of the underlying space (for simplicity take the whole space X). So the functor $\Gamma(X, -)$ taking global sections (on the category of sheaves of abelian groups say, where we have a notion of exact sequences) is not exact. Studying in which way exactness fails leads, by the general formalism of "derived functors", to the notion of cohomology groups $H^i(X, \mathscr{F})$ on X with coefficients in a sheaf $\mathscr{F}$ of abelian groups. (Often $\mathscr{F}$ will be a $\mathscr{O}_X$-module). This is an extremely powerful tool to "algebraize" geometric information. After this "controled loss of information" often things are easier to work with.

The Theorem of Riemann-Roch. (Theorem 3.13, Section (6.17)) This famous theorem is a specific example where cohomology of $\mathscr{O}_X$-modules attached to divisors on a smooth projective curve can be used very profitably in order to study the geometry of the curve. Among many other things it yields a very clean description of the group structure on an elliptic curve, giving us the associativity (which we had to leave open in Part 1) basically for free.

1. Proper morphisms

The functorial point of view.

(1.1) Schemes as functors.

References: [GW1] Sections (4.1), (4.2); [Mu] II.6.

As we have discussed in Algebraic Geometry 1, to a scheme X we can attach its functor of T-valued points:

$$h_X\colon (\mathrm{Sch})^{\mathrm{opp}} \to (\mathrm{Sets}), \quad T \mapsto X(T) := \mathrm{Hom}(T, X),$$

which on morphisms is just given by composition: $f\colon T' \to T$ is mapped under h_X to the map $X(T) \to X(T')$, $\alpha \mapsto \alpha \circ f$.

Example 1.1.

(1) For any affine scheme $X = \operatorname{Spec} A$, we have $X(T) = \mathrm{Hom}(A, \Gamma(T, \mathscr{O}_T))$. For example, this gives $\mathbb{A}^n_R(T) = \Gamma(T, \mathscr{O}_T)^n$ for any ring R and any R-scheme T (where we understand $\mathbb{A}^n_R(T)$ as the set of morphisms $T \to \mathbb{A}^n_R$ *of R-schemes*).

(2) It is more difficult to describe $\mathbb{P}^n(T)$ for general T (for T the spectrum of a field, we have the description by homogeneous coordinates). We will come back to this later.

Using the notion of a morphism of functors, we can speak of the category $\widehat{\mathscr{C}} := \mathrm{Func}((\mathrm{Sch})^{\mathrm{opp}}, (\mathrm{Sets}))$ of all such functors, and we obtain a functor $h\colon (\mathrm{Sch}) \to \widehat{\mathscr{C}}$, $X \mapsto h_X$, which on morphisms is – once again – defined by composition: $\alpha\colon X' \to X$ is mapped by h to the morphism $h_{X'} \to h_X$ of functors given by $X'(T) \to X(T)$, $\beta \mapsto \beta \circ \alpha$.

Even though at first sight this may look complicated, this is an entirely "formal" (i.e., category-theoretic) procedure which has nothing to do with schemes. In fact, if $\mathscr{C}$ is any category (as always, assumed to be *locally small*, i.e., each $\mathrm{Hom}(X, Y)$ is a set), for an object X of $\mathscr{C}$ we can define the functor

$$h_X\colon \mathscr{C}^{\mathrm{opp}} \to (\mathrm{Sets}), \quad T \mapsto X(T) := \mathrm{Hom}_{\mathscr{C}}(T, X),$$

and now setting $\widehat{\mathscr{C}} := \mathrm{Func}(\mathscr{C}^{\mathrm{opp}}, (\mathrm{Sets}))$, we obtain a functor $h\colon \mathscr{C} \to \widehat{\mathscr{C}}$.

Theorem 1.2. (Yoneda Lemma) *For any category $\mathscr{C}$, the functor h constructed above is fully faithful.*

Proof. Given X and Y and a morphism $\Phi\colon h_X \to h_Y$, we obtain a morphism $X \to Y$ by applying Φ to $\mathrm{id}_X \in h_X(X)$. One checks that this is an inverse of the map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(h_X, h_Y)$ given by h. $\square$

Remark 1.3. The Yoneda lemma says that the map $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}(h_X, h_Y)$ is a bijection for all X, Y. We may rewrite the left hand side as $h_Y(X)$. More generally, with basically the same argument one can show that for every functor $F\colon \mathscr{C}^{\mathrm{opp}} \to (\mathrm{Sets})$ there are identifications $F(X) = \mathrm{Hom}_{\mathrm{Func}(\mathscr{C}^{\mathrm{opp}},(\mathrm{Sets}))}(h_X, F)$ that are functorial in X.

We will mostly apply the Yoneda Lemma to the category of schemes, or the category of S-schemes for some fixed scheme S. Let us list some of its consequences (as before, these facts are not specific to the category of schemes):

(1) Let X, Y be schemes. The following are equivalent:
  (i) $X \cong Y$,
  (ii) $h_X \cong h_Y$ (isomorphism of functors)
  (iii) there exists a family $f_T\colon X(T) \to Y(T)$ of bijections of sets that is functorial in T, i.e., for every scheme morphism $T' \to T$, the diagram

$$\begin{array}{ccc} X(T) & \xrightarrow{f_T} & Y(T) \\ \downarrow & & \downarrow \\ X(T') & \xrightarrow{f_T} & Y(T') \end{array}$$

  is commutative.

(2) Let X, Y be schemes. Giving a scheme morphism $X \to Y$ is equivalent to giving a family of maps $f_T\colon X(T) \to Y(T)$ of sets for each scheme T, that is functorial in T (same condition as in (1) (iii)). *Example.* The determinant of a matrix is a scheme morphism $\mathbb{A}^{n^2} \to \mathbb{A}^1$.

(3) A diagram of scheme morphisms is commutative if and only if for every scheme T the diagram (in the category of sets) obtained by replacing each scheme by its set of T-valued points, and replacing the scheme morphisms by the induced maps of sets, is commutative.

Given a functor $F\colon \mathscr{C}^{\mathrm{opp}} \to (\mathrm{Sets})$, it "usually" will not be isomorphic to a functor of the form h_X. If it is, it is called *representable* (by X), and this is a very special property. On the other hand, especially in algebraic geometry, i.e., when $\mathscr{C}$ is the category of schemes (or of S-schemes for some fixed scheme S), it turns out that many naturally appearing functors are in fact representable. Proving such representability results is often quite difficult, but at the same time extremely useful, and this approach yields some of the most interesting examples of schemes.

Fiber products and base change.

(1.2) Recap: fiber products of schemes.

References: [GW1] Sections (4.4)–(4.6); [H] II.3; [Mu] II.2.

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The universal property of a *fiber product* generalizes the universal property of a product (of two objects, in any category). It is defined as follows. (See the lecture notes for Algebraic Geometry 1 for a bit more background, e.g., an explanation of the term *fiber product*.)

Definition 1.4. *Let $\mathscr{C}$ be a category. We fix morphisms $f\colon X \to S$ and $g\colon Y \to S$. Then an object P in $\mathscr{C}$ together with morphisms $p\colon P \to X$ and $q\colon P \to Y$ such that $f \circ p = g \circ q$ is called a* fiber product *of X and Y over S, if for every object T of $\mathscr{C}$ together with morphisms $\alpha\colon T \to X$ and $\beta\colon T \to Y$ with $f \circ \alpha = g \circ \beta$, there exists a unique morphism $\xi\colon T \to P$ with $p \circ \xi = \alpha$, $q \circ \xi = \beta$.*

We say that a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is *cartesian*, if it is a fiber product diagram, i.e., if A satisfies the universal property defining the fiber product of B and C over D.

Theorem 1.5.

(1) *If $f\colon X \to S$, $g\colon Y \to S$ are morphisms of schemes, then the fiber product of X and Y over S exists.*

(2) *If in (1) $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} R$ are affine schemes (so that f and g are given by ring homomorphisms $R \to A$, $R \to B$, then $\operatorname{Spec} A \otimes_R B$ together with the morphisms induced by the natural maps $A \to A \otimes_R B$, $B \to A \otimes_R B$, is the fiber product of X and Y over S.*

(3) *For f, g as in (1), and open covers $S = \bigcup_i U_i$, $f^{-1}(U_i) = \bigcup_j V_{ij}$, $g^{-1}(U_i) = \bigcup_k W_{ik}$, for all i, j, k, the natural morphism $V_{ij} \times_{U_i} W_{ik} \to X \times_S Y$ induced by the universal property of the fiber product is an open immersion, and taken together the open subschemes of the above form cover $X \times_S Y$.*

Example 1.6. If $f\colon X \to S$ is any morphism and $V \to S$ is an open immersion, then we can identify $X \times_S V = f^{-1}(V)$ as X-schemes, i.e., the projection $X \times_S V \to X$ is an open immersion which induces an isomorphism $X \times_S V \cong f^{-1}(V)$ (where we view $f^{-1}(V)$ as an open subscheme of X).

Example 1.7. (Fibers of a morphism) If $f\colon X \to S$ is a morphism of schemes and $s \in S$, then we call the fiber product $X_s := f^{-1}(s) := X \times_S \operatorname{Spec} \kappa(s)$ (with respect to f and the natural morphism $\operatorname{Spec} \kappa(s) \to S$) the *scheme-theoretic fiber of f over s*. The projection $X_s \to X$ induces a homeomorphism between the topological space of the scheme X_s and the fiber of the continuous map f over s. Cf. the Algebraic Geometry 1 class, or see [GW1] Section (4.8).

Lemma 1.8. *In the list below, the "obvious" maps between fiber products are isomorphisms:*

(1) $X \times_S S \cong X$,

(2) $X \times_S Y \cong Y \times_S X$,

(3) $(X \times_S Y) \times_T Z \cong X \times_S (Y \times_T Z)$ *(and this allows us to omit the parentheses in expressions like these).*

(4) *Let* $f\colon X \to Y$ *be a morphism and let* $\Gamma_f\colon X \xrightarrow{(\mathrm{id},f)} X \times_S Y$ *be the graph of* f*. Then the following diagram is a fiber product diagram:*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \big\downarrow{\scriptstyle \Gamma_f} & & \big\downarrow{\scriptstyle \Delta} \\ X \times_S Y & \xrightarrow{f\times \mathrm{id}} & Y \times_S Y. \end{array}$$

(These claims have to be read in the "natural way", e.g., in (1) the fiber product on the left is formed with respect to $\mathrm{id}\colon S \to S$, and in (3) morphisms $X \to S$, $Y \to S$, $Y \to T$, $Z \to T$ are given and used to form the fiber products, and the morphism $X \times_S Y \to T$ is the composition $X \times_S Y \to Y \to T$ of the projection to the second factor with the given morphism $Y \to T$, and similarly on the right hand side.)

Proof. These properties can easily be checked using the universal property (or, what more or less amounts to the same, by the Yoneda lemma). In any case, this reduces to checking the above claims for fiber products of sets, where they follow immediately from the explicit description of fiber products of sets. □

Example 1.9. (Group schemes) Let S be a scheme. A *group scheme over* S is an S-scheme G together with a functor $h\colon ((\mathrm{Sch})/S)^{\mathrm{opp}} \to (\mathrm{Grp})$, such that h_G is the composition of h and the forgetful functor $(\mathrm{Grp}) \to (\mathrm{Sets})$. In other words, for every S-scheme T, we are given a group structure on $G(T)$, and for every morphism $T' \to T$, the induced map $G(T) \to G(T')$ is a group homomorphism.

In view of the above discussion, we can express this structure equivalently by giving a multiplication morphism $m\colon G \times_S G \to G$, a morphism $i\colon G \to G$ ("inverse element") that induces the map $g \mapsto g^{-1}$ on each $G(T)$, and a morphism $S \to G$ ("neutral element") that induces the neutral element in each $G(T)$ (note that for every S-scheme T, the set $S(T)$ is a singleton). The morphisms m, i, e have to satisfy certain conditions reflecting the group axioms; the conditions can be expressed by requiring that certain diagrams be commutative. See [GW1] Section (4.15).

Some examples: Fix an affine scheme $S = \mathrm{Spec}(R)$. We will consider group schemes over S.

(1) The trivial group scheme $G = S$. Then $G(T)$ is a singleton set for every S-scheme T, and we give it the structure of the trivial group.

(2) The *additive group* $\mathbb{G}_a$ (over S). As a scheme, we define it as $\mathbb{G}_a = \mathrm{Spec}\, R[X]$. Then for every S-scheme T we have, since $\mathbb{G}_a$ is affine,

$$\mathbb{G}_a(T) = \mathrm{Hom}_S(T, \mathbb{G}_a) = \mathrm{Hom}_R(R[X], \Gamma(T, \mathscr{O}_T)) = \Gamma(T, \mathscr{O}_T)$$

and we equip it with the group structure given by the addition of the ring $\Gamma(T, \mathscr{O}_T)$.

(3) The *multiplicative group* $\mathbb{G}_m$ (over S). As a scheme, we define it as $\mathbb{G}_m = \operatorname{Spec} R[X, X^{-1}]$. Then for every S-scheme T we have, since $\mathbb{G}_m$ is affine,

$$\mathbb{G}_m(T) = \operatorname{Hom}_S(T, \mathbb{G}_m) = \operatorname{Hom}_R(R[X, X^{-1}], \Gamma(T, \mathscr{O}_T)) = \Gamma(T, \mathscr{O}_T)^\times$$

and we equip it with the multiplicative group structure inherited from the multiplication of the ring $\Gamma(T, \mathscr{O}_T)$.

(4) The *general linear group* GL_n (as a group scheme over S). This should be a group scheme GL_n such that for every S-scheme T we have

$$GL_n(T) = GL_n(\Gamma(T, \mathscr{O}_T)),$$

where the right hand side is the “usual” general linear group of invertible matrices with entries in the ring $\Gamma(T, \mathscr{O}_T)$. Clearly this is a functorial group structure on all these sets, but we still have to justify that there exists an S-scheme that has these sets as its T-valued points. But this is not difficult: We view the affine space $\mathbb{A}_S^{n^2} = \operatorname{Spec}(R[X_{i,j};\ i, j = 1, \dots, n])$ of dimension n^2 as the space of $n \times n$-matrices, i.e.,

$$\mathbb{A}_S^{n^2}(T) = \Gamma(T, \mathscr{O}_T)^{n^2} = \operatorname{Mat}_n(\Gamma(T, \mathscr{O}_T)),$$

denote by $\det \in R[X_{i,j};\ i, j = 1, \dots, n]$ the determinant of the matrix $(X_{i,j})_{i,j}$ given by the indeterminates, and define $GL_n = D(\det)$, the principal open subscheme defined by det, as a scheme. This has the desired T-valued points.

(5) Elliptic curves also are an (interesting!) example, but what we have done in Part 1 of the lecture is not a full proof that they carry a group scheme structure (but could be made into one without too much work: one would have to express the group law that we have constructed on $E(k')$ for every extension field k' of the base field k in terms of explicit formulas and use those to show that the multiplication and the rule mapping each element to its inverse (=negative) are in fact given by scheme morphisms).

(1.3) Base change.

References: [GW1] Chapter 4, in particular Sections (4.7)–(4.10).

Given scheme morphisms $f\colon X \to S$ and $g\colon S' \to S$, we call the projection $X \times_S S' \to S'$ the *morphism obtained from f by base change along g*. This defines a functor from the category of S-schemes to the category of S'-schemes. April 21, 2026

A particularly simple example is the case where $g\colon V \to S$ is an open immersion. In that case the base change of f is just the restriction of f to $f^{-1}(V) \to V$.

Many properties of scheme morphisms are “stable under base change” in the following sense: A property **P** of scheme morphisms is called stable under base change if for every morphism $f\colon X \to Y$ of S-schemes that has property $\mathbf{P}$ and every scheme morphism $S' \to S$, the induced morphism $X \times_S S' \to Y \times_S S'$ also has property $\mathbf{P}$.

Given a property $\mathbf{P}$, to check that it is stable under base change, it is enough to check that whenever $f\colon X \to S$ has the property, and $g\colon S' \to S$ is a scheme morphism, then $X \times_S S' \to S'$ also has the property. In fact, this is clearly a special case of the above definition (namely the case where $Y = S$). On the other hand, suppose this special case is true and $f\colon X \to Y$ is any morphism of S-schemes. Identifying $X \times_S S' = X \times_Y (Y \times_S S')$ using the rules of "computations with fiber products" (Lemma 1.8), the base change $X \times_S S' \to Y \times_S S'$ is identified with the projection $X \times_Y (Y \times_S S') \to Y \times_S S'$. Applying the special case to $X \to Y$ and the base change $Y \times_S S' \to Y$, we obtain that $X \times_S S' = X \times_Y (Y \times_S S') \to Y \times_S S'$ has property $\mathbf{P}$.

Proposition 1.10. *The following properties of scheme morphisms are stable under base change: Being …*

(1) *an isomorphism,*
(2) *an open immersion,*
(3) *a closed immersion,*
(4) *an immersion,*
(5) *surjective,*
(6) …most of the properties of scheme morphisms that we will get to know later in the course …

A notable exception is the property of being injective: Can you find an example of an injective morphism $X \to S$ of schemes and a morphism $S' \to S$ such that the base change $X \times_S S' \to S'$ is not injective?

All the properties in the above list, and also being injective, are *stable under composition*, i.e., if two composable morphisms both have the property, then so does the composition.

Example 1.11.

(1) If $R \to R'$ is a ring homomorphism, then $\mathbb{A}^n_R \otimes_R R' := \mathbb{A}^n_R \times_{\operatorname{Spec} R} \operatorname{Spec} R' = \mathbb{A}^n_{R'}$. In view of this we define, for an arbitrary scheme S, $\mathbb{A}^n_S := \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} S$.
(2) If $R \to R'$ is a ring homomorphism, then $\mathbb{P}^n_R \otimes_R R' := \mathbb{P}^n_R \times_{\operatorname{Spec} R} \operatorname{Spec} R' = \mathbb{P}^n_{R'}$. In view of this we define, for an arbitrary scheme S, $\mathbb{P}^n_S := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} S$.

Remark 1.12. Recall the notion of separated morphism: $Y \to S$ is called separated, if the diagonal morphism $\Delta\colon Y \to Y \times_S Y$ is a *closed* immersion (it is always an immersion, and the question really is whether the topological image is a closed subspace). This is a scheme-theoretic analogue of the Hausdorff property of topolgical spaces.

As an application of the above we obtain the following. Let $Y \to S$ be separated and let $f\colon X \to Y$ be any morphism. Then the graph $\Gamma_f\colon X \to$

$X \times_S Y$ is a closed immersion, because being a closed immersion is stable under base change and because of the cartesian diagram in Lemma 1.8 (4).

Proper morphisms.

(1.4) Proper maps between topological spaces.

References: [Bou-TG] Ch. I §10, [Stacks] Section 005M.

Most schemes that we have encountered so far (in particular, all affine schemes, projective space over any ring, subschemes $V_+(I)$ of projective space over a ring, ...) are quasi-compact. On the other hand, from a geometric point of view, e.g., the affine line (or higher-dimensional affine space) "should not be viewed" as a compact space. The notion of properness is a suitable replacement in algebraic geometry for the notion of compactness in topology/differential geometry.

Similarly as separatedness, we will define properness in terms of fiber products of schemes, starting from a characterization of quasi-compact topological spaces, given by the notion of *proper map* between continuous spaces, which we discuss below as a motivation for the definition of proper scheme morphisms. The purpose of motivation aside, the rest of this section plays no role in the course.

Note that fiber products in the category of topological spaces exist. In fact, for continous maps $X \to S$, $Y \to S$, the set-theoretic fiber product $X \times_S Y$, equipped with the subspace topology for the inclusion $X \times_S Y \subseteq X \times Y$ (where the right hand side carries the product topology) is easily seen to satisfy the required universal property.

Recall that a continous map $f\colon X \times Y$ is *closed*, if for every closed subset $C \subseteq X$, the image $f(C) \subseteq Y$ is closed.

Definition 1.13.

(1) *We call a continuous map $f\colon X \to Y$* universally closed*, if for every continuous map $Z \to Y$, the "base change" of f along $Z \to Y$, i.e., the induced map $X \times_Y Z \to Z$, is closed.*

(2) *We call a continuous map $f\colon X \to Y$* Bourbaki-proper*, if for every topological space Z, the induced map $f \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$ is closed.*

Proposition 1.14. *Let $f\colon X \to Y$ be a continuous map. The following properties are equivalent.*

(i) *f is universally closed,*

(ii) *f is Bourbaki-proper,*

(iii) *f is closed and for every $y \in Y$ the fiber $f^{-1}(y)$ is quasi-compact.*

(iv) *f is closed and for every quasi-compact subset $K \subseteq Y$, the inverse image $f^{-1}(K)$ is quasi-compact.*

See [Stacks] Theorem 005R.

(1.5) Morphisms of finite type.

References: [GW1] Sections (10.1), (10.2).

To define proper morphisms of schemes in the following section, we also need the following ingredients.

Definition 1.15. *A morphism $f\colon X \to Y$ is called* quasi-compact*, if for every quasi-compact open $V \subseteq Y$ the inverse image $f^{-1}(V)$ is quasi-compact.*

Lemma 1.16. *Let $f\colon X \to Y$ be a morphism of schemes. The following are equivalent.*
(i) *The morphism f is quasi-compact.*
(ii) *For every affine open subscheme $V \subseteq Y$, the inverse image $f^{-1}(V)$ is quasi-compact.*
(iii) *There exists a cover $Y = \bigcup_i V_i$ by affine open subschemes such that for every i the inverse image $f^{-1}(V_i)$ is quasi-compact.*

Note that in part (iii) of the lemma it is important to consider a cover by *affine* open subschemes.

Recall that an algebra B over a ring A is called *of finite type* (or equivalently, *finitely generated*) if there exists $n \geq 0$ and a surjective A-algebra homomorphism $A[X_1, \dots, X_n] \to B$. We then also say that the ring homomorphism $A \to B$ is of finite type. For example, if R is any ring, $f \in R$, then the homomorphism $R \to R_f$ is of finite type, since $R_f \cong R[X]/(fX - 1)$. (On the other hand, if $\mathfrak{p} \subset R$ is a prime ideal, then the homomorphism $R \to R_{\mathfrak{p}}$ typically is not of finite type.) If $A \to B$ and $B \to C$ are ring homomorphisms of finite type, then the composition $A \to C$ is also of finite type.

Definition 1.17.
(1) *A morphism $f\colon X \to Y$ of schemes is called* locally of finite type *(or: X is called a Y-scheme locally of finite type, of locally of finite type over Y), if for every affine open subscheme $V \subseteq Y$ and every open subscheme $U \subseteq f^{-1}(V)$, the ring homomorphism $\Gamma(V, \mathscr{O}_Y) \to \Gamma(U, \mathscr{O}_X)$ induced by the restriction $U \to V$ of f makes $\Gamma(U, \mathscr{O}_X)$ a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type.*
(2) *A morphism $f\colon X \to Y$ of schemes is called* of finite type *(or: X is called a Y-scheme of finite type, or of finite type over Y), if f is locally of finite type and quasi-compact.*

Lemma 1.18. *Let $f\colon X \to Y$ be a morphism of schemes. The following are equivalent.*
(i) *The morphism f is locally of finite type.*
(ii) *There exist a cover $Y = \bigcup_i V_i$ by affine open subschemes, and for each i a cover $f^{-1}(V_i) = \bigcup_j U_{ij}$ by affine open subschemes such that for all i, j the $\Gamma(V_i, \mathscr{O}_Y)$-algebra $\Gamma(U_{ij}, \mathscr{O}_X)$ is of finite type.*

Proof. This kind of statement holds for many interesting properties of morphisms of schemes. While the proof does not use any difficult input, it is quite long; but it nicely illustrates how to work with schemes and to eventually reduce scheme-theoretic statements to commutative algebra.

Step 1. We first show that without loss of generality, we may assume $V = Y$. In fact, if $D_{V_i}(s) \subset V_i$ is a principal open subset, then $f^{-1}(D_{V_i}(s)) = \bigcup_j D_{U_{ij}}(s_j)$, where $s_j \in \Gamma(U_{ij}, \mathscr{O}_X)$ is the image of s, and the $\Gamma(D_{V_i}(s), \mathscr{O}_Y)$-algebra $\Gamma(D_{U_{ij}}(s_j), \mathscr{O}_X)$ is of finite type. Thus replacing the V_i by suitable principal open subsets, we may assume that V is a union of some of the V_i, and can thus replace Y by V. Moreover, we may choose these principal opens in V_i so that at the same time they are *principal* open in V (Lemma 1.19 below), so we may assume in addition that each V_i is principal open in Y.

Step 2. We may replace U_{ij} by a cover by principal opens, and thus assume that U is covered by some of the U_{ij}. Moreover, as in Step 1 we may choose these principal opens in U_{ij} so that at the same time they are *principal* open in U. We may therefore assume, without loss of generality, that $U = X$ and that every U_{ij} is principal open in X.

Step 3. We now have the following situation: $X \to Y$ is a morphism of affine schemes, say corresponding to a ring homomorphism $A \to B$. We want to show that $A \to B$ is of finite type. We have a covering $Y = \bigcup V_i$ by principal open subsets (which we may assume to be finite, since Y is quasi-compact), and coverings $f^{-1}(V_i) = \bigcup_j U_{ij}$ (which likewise we may assume to be finite) by principal opens of X, such that the homomorphism $\Gamma(V_i, \mathscr{O}_Y) \to \Gamma(U_{ij}, \mathscr{O}_X)$ is of finite type. Since $\Gamma(V, \mathscr{O}_Y) \to \Gamma(V_i, \mathscr{O}_Y)$ is also of finite type and this property is stable under composition, we also have that $\Gamma(V, \mathscr{O}_Y) \to \Gamma(U_{ij}, \mathscr{O}_X)$ is of finite type. We may therefore “forget about the V_i” and simply apply Lemma 1.20 to $A \to B$ and the cover $\mathrm{Spec}(B) = \bigcup_{i,j} U_{ij}$. □

In the proof we have used the following lemmas.

Lemma 1.19. *Let X be a scheme, $x \in X$, and U, V affine open subschemes of X which both contain x. Then there exists an open neighborhood $W \subseteq U \cap V$ of x which is principal open both in U and in V.*

Proof. One easily checks that a principal open in a principal open of some affine scheme U is principal open in U. Thus after replacing U by a suitable principal open, we may assume $U \subseteq V$. Now let $s \in \Gamma(V, \mathscr{O}_X)$ such that $D_V(s) \subseteq U \cap V$. This has the desired property since $D_V(s) = D_U(s_{|U})$. (By assumption, $D_V(s) \subseteq U$, and we can check whether a point of U lies in this set by checking whether the image of s in the residue class field is $\neq 0$, and the residue class field does not depend on whether we compute it inside U or V.) □

Lemma 1.20. *Let $A \to B$ be a ring homomorphism, and let $b_1, \dots, b_r \in B$ be a family of elements which generates the unit ideal and such that for every i the homomorphism $A \to B_{b_i}$ is of finite type. Then $A \to B$ is of finite type.*

Proof. For each i let $\left(\frac{c_{ij}}{b_i^{\nu_j}}\right)_j$ be a finite system of generators of B_{b_i} as an A-algebra, where $c_{ij} \in B$. Write $1 = \sum a_i b_i$. Let $B' \subseteq B$ be the A-subalgebra generated by all b_i, all a_i and all c_{ij}. We will show that $B' = B$, thus proving the lemma. So let $b \in B$. Its image in each B_{b_i} can be written as a polynomial expression in the $\frac{c_{ij}}{b_i^{\nu_j}}$ with coefficients in A, and clearing denominators, we find $N \geq 0$ such that $b_i^N b \in B'$. We may assume that the same N works for every i. Since B' contains the elements a_i, the b_i generate the unit ideal in B', so the same is true for the b_i^N. Therefore the above implies that $b \in B'$, as desired. □

Each of the properties of being *locally of finite type*, *quasi-compact*, and *of finite type* is stable under composition and under base change.

(1.6) Proper morphisms.

References: [GW1] Section (12.13); [H] II.4; [Mu] II.7.

Definition 1.21. *Let $f\colon X \to Y$ be a morphism of schemes.*

(1) *The morphism f is called* closed*, if for every closed subset $Z \subseteq X$, the image $f(Z)$ is a closed subset of Y.*
(2) *The morphism f is called* universally closed*, if for every morphism $Y' \to Y$ the base change $X \times_Y Y' \to Y'$ of f along Y' is a closed morphism.*
(3) *The morphism f is called* proper*, if it is separated, of finite type, and universally closed.*

Example 1.22.

(1) The affine line is not proper. More precisely, let k be a field, let $Y = \operatorname{Spec}(k)$, and let $X = \mathbb{A}^1_k$. Let $f\colon X \to Y$ be the natural morphism. Then f is separated, of finite type and closed, but (why?) not universally closed.
(2) Every closed immersion is proper.
(3) The property of being proper is stable under composition and under base change.

(1.7) Projective schemes are proper.

Definition 1.23. *Let S be a scheme. An S-scheme X is called* projective *(we also say that X is projective over S, or that the morphism $X \to S$ is projective), if there exist $N \geq 0$ and a closed immersion $X \hookrightarrow \mathbb{P}^N_S$ of S-schemes.*

This definition of *projective schemes* differs slightly from the one in [GW1] (Definition 13.68, which requires only that the above property holds locally on S). If S is affine, they coincide, however, and the difference will not be of any concern for us in this course. See [GW1], Summary 13.71 for a discussion. The definition given here is the one used in [H] and in [Stacks].

Suppose that $S = \operatorname{Spec} R$ is an affine scheme. For any homogeneous ideal $I \subseteq R[X_0, \dots, X_N]$, $V_+(I)$ is a closed subscheme of $\mathbb{P}^N_S$, and hence in particular a projective S-scheme. One can show that for S affine every projective scheme is isomorphic to a scheme of this form.

We will study this notion in more detail later (see Chapter 5).

Before we come to the main theorem of this section (Theorem 1.27), recall that for a homogeneous ideal $I \subseteq R[X_0, \dots, X_n]$ (where R is some ring) we have defined a closed subscheme $V_+(I)$ of $\mathbb{P}^n_R$. We need the following two results on closed subschemes of projective space.

Lemma 1.24. *Let R be a ring, $n \geq 0$, and let $I \subseteq (X_0, \dots, X_n) \subseteq R[X_0, \dots, X_n]$ be a homogeneous ideal. Then $V_+(I) = \emptyset$ if and only if $\operatorname{rad}(I) \supseteq (X_0, \dots, X_n)$.*

Proposition 1.25. ([GW1] Proposition 13.24) *Let R be a ring, $n \geq 0$, and let $Z \subseteq \mathbb{P}^n_R$ be a closed subscheme. Then there exists a homogeneous ideal $I \subseteq R[X_0, \dots, X_n]$ such that $Z \cong V_+(I)$.*

In addition, we will use the following commutative algebra lemma which is easily proved using the definitions of the localizations appearing in the lemma. (We will later generalize the lemma when we prove that given an $\mathscr{O}_X$-module $\mathscr{F}$ of finite type on a locally ringed space X, the support of $\mathscr{F}$, i.e., the set of all points x such that the stalk $\mathscr{F}_x$ does not vanish, is closed. See Proposition 2.18.)

Lemma 1.26. *Let R be a ring, let $\mathfrak{p} \subset R$ be a prime ideal and let M be a finitely generated R-module. If the localization $M_\mathfrak{p}$ vanishes, then there exists $s \in R \setminus \mathfrak{p}$ such that already the localization M_s is zero.*

Theorem 1.27. ([GW1] Theorem 13.40) *Every projective morphism of schemes is proper.*

Sketch of proof. Since closed immersions are proper, it is enough to prove that projective space is proper, i.e., that for every scheme S the morphism $\mathbb{P}^n_S \to S$ is closed. Since this property can be checked locally on S, we may assume that $S = \operatorname{Spec} R$ is affine.

If $Z \subseteq \mathbb{P}^n_S$ is a closed subset, there exists a closed subscheme with underlying topological space Z, and hence (Proposition 1.25) a homogeneous ideal $I \subseteq R[X_0, \dots, X_n]$ such that $V_+(I)$ has underlying topological space Z. We need to show that the image of $V_+(I)$ in S is closed, or equivalently, that its complement $U \subseteq S$ is open.

Denote by f the composition $V_+(I) \hookrightarrow \mathbb{P}^n_S \to S$, and let $x \in U$. Then the scheme-theoretic fiber $f^{-1}(x) = V_+(I) \times_U \operatorname{Spec} \kappa(x)$ is empty. We want to show that there exists $s \in R$ such that $x \in D(s) \subseteq U$. The inclusion $D(s) \subseteq U$ amounts to saying that $f^{-1}(D(s)) = \emptyset$. To translate the problem into a commutative algebra statement, let $\overline{I}$ be the image of I in $\kappa(x)[X_0,\dots,X_n]$. It follows from the assumption $f^{-1}(x)=\emptyset$ and Lemma 1.24 that $\mathrm{rad}(\overline{I})$ contains the ideal $(X_0,\dots,X_n)$ ($\subset \kappa(x)[X_0,\dots,X_n]$). Thus for d sufficiently large, for the degree d components we have $\overline{I}_d=(X_0,\dots,X_n)_d$. By the lemma of Nakayama, we obtain $I_d\otimes \mathscr{O}_{S,x}=(X_0,\dots,X_n)_d\subseteq \mathscr{O}_{S,x}[X_0,\dots,X_n]$. It then follows that the analogous equality holds already over the localization of R with respect to a suitable element s not contained in the prime ideal x. □

Example 1.28. (The resultant of polynomials) Let k be an algebraically closed field (with some "obvious" adaptations, the results below hold over an arbitrary field). Let $m,n\in\mathbb{N}$. We identify the set A of pairs (f,g) of monic polynomials with the set of k-valued points of the affine space $\mathbb{A}_k^{m+n}=\operatorname{Spec} k[S_0,\dots,S_{m-1},T_0,\dots,T_{n-1}]$, where a tuple $(s_i,t_j)\in k^{m+n}=\mathbb{A}_k^{m+n}(k)$ corresponds to $\big(X^m+s_{m-1}X^{m-1}+\dots+s_0, X^n+t_{n-1}X^{n-1}+\dots+t_0\big)$.

Viewing A as the set of closed points of $\mathbb{A}_k^{m+n}$, A is equipped with a topology, namely the topology induced by the Zariski topology.

Let $Z\subset A$ be the subset consisting of those pairs (f,g) such that f and g have a common zero in k. Write $R=k[S_0,\dots,S_{m-1},T_0,\dots,T_{n-1}]$.

Claim. The set Z is a closed subset.

Proof of claim. Let

$$F=X^m+S_{m-1}X^{m-1}+\dots+S_1X+S_0,$$
$$G=X^n+T_{n-1}X^{n-1}+\dots+T_1X+T_0\in R[X]$$

be the "universal" monic polynomials, and let

$$\tilde{F}=X^m+S_{m-1}X^{m-1}Y+\dots+S_1XY^{m-1}+S_0Y^m,$$
$$\tilde{G}=X^n+T_{n-1}X^{n-1}Y+\dots+T_1XY^{n-1}+T_0Y^n\in R[X,Y]$$

be their homogenizations with respect to a second variable Y.

Let $p\colon \mathbb{P}^1_R\to\operatorname{Spec} R$ be the projection. Then $Z=p(V_+(\tilde{F},\tilde{G}))\cap A$. By the above theorem, $p(V_+(\tilde{F},\tilde{G}))$ is closed in $\mathbb{A}_k^{m+n}$, hence the claim follows. To see the equality, fix $x=(f,g)\in A$ and let $\tilde{f},\tilde{g}$ be their homogenizations. Then f and g have a common zero in k if and only if $V_+(\tilde{f},\tilde{g})\neq\emptyset$ (inside $\mathbb{P}^1_{\kappa(x)}$). Note that the point $(1:0)$, the "point at infinity" in $\mathbb{P}^1$ is never a zero of $\tilde{f}$ or $\tilde{g}$. Since $V_+(\tilde{f},\tilde{g})=V_+(\tilde{F},\tilde{G})\times_{\operatorname{Spec} R}\operatorname{Spec}\kappa(x)$ can be identified with the (scheme-theoretic) fiber of p over the point x, this proves the desired description of Z.

More precisely one can show (using other methods) that Z is the zero locus of a single polynomial in R, the so-called *resultant* of a pair of monic polynomials. See [GW1] Section (B.20) for a sketch and further references, or [Bo] Abschnitt 4.4 for a detailed account in German.

2. $\mathscr{O}_X$-modules

General references: [GW1] Ch. 7, [H] II.5.

Definition and basic properties.

(2.1) Definition of $\mathscr{O}_X$-modules.

Definition 2.1. *Let $(X, \mathscr{O}_X)$ be a ringed space. An $\mathscr{O}_X$-module is a sheaf $\mathscr{F}$ of abelian groups on X together with maps*

$$\mathscr{O}_X(U) \times \mathscr{F}(U) \to \mathscr{F}(U) \qquad \text{for each open } U \subseteq X$$

giving each $\mathscr{F}(U)$ the structure of an $\mathscr{O}_X(U)$-module, and which are compatible with the restriction maps for open subsets $U' \subseteq U \subseteq X$.

An $\mathscr{O}_X$-module homomorphism $\mathscr{F} \to \mathscr{G}$ between $\mathscr{O}_X$-modules $\mathscr{F}$, $\mathscr{G}$ on X is a sheaf morphism $\mathscr{F} \to \mathscr{G}$ such that for all open subsets $U \subseteq X$, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is a homomorphism of $\mathscr{O}_X(U)$-modules. We denote the set of $\mathscr{O}_X$-module homomorphisms from $\mathscr{F}$ to $\mathscr{G}$ by $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$; this is an $\mathscr{O}_X(X)$-module (and in particular an abelian group).

We obtain the category ($\mathscr{O}_X$-Mod) of $\mathscr{O}_X$-modules.

Remark 2.2. If $\mathscr{F}$ is an $\mathscr{O}_X$-module and $x \in X$, then the stalk $\mathscr{F}_x$ carries a natural $\mathscr{O}_{X,x}$-module structure. The $\kappa(x)$-vector space $\mathscr{F}(x) := \mathscr{F}_x \otimes_{\mathscr{O}_{X_x}} \kappa(x)$ is called the *fiber of $\mathscr{F}$ over x*.

Constructions, examples 2.3. Let X be a ringed space, $\mathscr{F}$ an $\mathscr{O}_X$-module.

(1) $\mathscr{O}_X$,
(2) submodules and quotients,
(3) $\oplus$, $\prod$, $- \otimes_{\mathscr{O}_X} -$, (filtered) colimits,
(4) kernels, cokernels, image, exactness; these are compatible with passing to the stalks, and exactness can be checked on stalks,
(5) restriction to open subsets: $\mathscr{F}_{X|U}$, $U \subseteq X$ open,
(6) The Hom sheaf $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$, defined by $U \mapsto \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}_{|U}, \mathscr{G}_{|U})$ (this is a sheaf, by "gluing of morphisms of sheaves"), duals: $\mathscr{F}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$.

The principle for most of these constructions is the following: Use the corresponding construction for modules over a ring for sections on opens of X, and then sheafify, if necessary. For products (and hence also for finite direct sums), kernels, and the Hom sheaf, the sheafification step is not required. For quotients, infinite direct sums, tensor products, colimits, cokernels and images, in general the presheaf obtained from the corresponding construction for modules is not a sheaf, so one has to sheafify.

The operations of taking kernels, cokernels, images, direct sums, tensor products, colimits are compatible with passing to stalks. (For products and the $\mathscr{H}om$ sheaf, this does not hold in general. But compare Proposition 2.17/Problem 15.)

The category of $\mathscr{O}_X$-modules is an abelian category.

Definition 2.4. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module on the ringed space X. We call $\mathscr{F}$*

(a) free*, if it is isomorphic to $\bigoplus_{i\in I} \mathscr{O}_X$ for some set I,*

(b) locally free*, if there exists an open covering $X = \bigcup_j U_j$ of X such that $\mathscr{F}_{|U_j}$ is a free $\mathscr{O}_{U_j}$-module for each j.*

The rank *of a free $\mathscr{O}_X$-module is the cardinality of I as above (we usually regard it in $\mathbb{Z} \cup \{\infty\}$, without making a distinction between infinite cardinals). The* rank *of a locally free $\mathscr{O}_X$-module is a function $X \to \mathbb{Z} \cup \{\infty\}$ which is locally constant on X (i.e., on each connected component of X, there is an integer giving the rank).*

An invertible sheaf *or* line bundle *on X is a locally free sheaf of rank* 1*.*

For $\mathscr{L}$ invertible, there is a natural isomorphism $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}^\vee \cong \mathscr{O}_X$ (whence the name), cf. Problem 14. Hence $\otimes$ induces a group structure on the set of isomorphism classes of invertible sheaves in X. The resulting group is called the Picard group of X and denoted by $\operatorname{Pic}(X)$.

(2.2) Inverse image.

Definition 2.5. *Let $f\colon X \to Y$ be a morphism of ringed spaces, and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then $f_*\mathscr{F}$ carries a natural $\mathscr{O}_Y$-module structure and is called the* direct image *or* push-forward *of $\mathscr{F}$ under f.*

Definition 2.6. *Let $f\colon X \to Y$ be a morphism of ringed spaces, $\mathscr{F}$ an $\mathscr{O}_Y$-module.*

We define

$$f^*\mathscr{F} := f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

For $x \in X$, we have $(f^*\mathscr{F})_x \cong \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x}$.

We obtain functors f_*, f^* between the categories of $\mathscr{O}_X$-modules and $\mathscr{O}_Y$-modules.

Proposition 2.7. *Let $f\colon X \to Y$ be a morphism of ringed spaces. The functors f_* is right adjoint to the functor f^*:*

$$\operatorname{Hom}_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \cong \operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F})$$

for all $\mathscr{O}_X$-modules $\mathscr{F}$, all $\mathscr{O}_Y$-modules $\mathscr{G}$, functorially in $\mathscr{F}$ and $\mathscr{G}$.

Quasi-coherent $\mathscr{O}_X$-modules.

(2.3) The $\mathscr{O}_{\operatorname{Spec} A}$-module attached to an A-module M.

Definition 2.8. *Let A be a ring and M an A-module. Then setting*

$$D(f) \mapsto M_f, \quad f \in A,$$

is well-defined and defines a sheaf on the basis of principal open sets in $\operatorname{Spec} A$. *We denote the corresponding sheaf on* $\operatorname{Spec} A$ *by $\widetilde{M}$. It is an $\mathscr{O}_{\operatorname{Spec} A}$-module (by viewing each M_f as an A_f-module in the natural way).*

In the situation of the definition, the stalk of $\widetilde{M}$ at a point $\mathfrak{p} \in \operatorname{Spec} A$ is the localization $M_{\mathfrak{p}}$.

Remark 2.9. For an affine scheme X, in general not every $\mathscr{O}_X$-module has the above form. We will investigate this more closely soon.

Proposition 2.10. *Let A be a ring, and let M, N be A-modules. Then the maps*

$$\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\mathscr{O}_{\operatorname{Spec} A}}(\widetilde{M}, \widetilde{N})$$

given by

$$\varphi \mapsto \widetilde{\varphi} := (\varphi_f \colon M_f \to N_f)_f$$

and, in the other direction,

$$\Phi \mapsto \Gamma(\operatorname{Spec} A, \Phi),$$

are inverse to each other. In other words, $\widetilde{\cdot}$ is a fully faithful functor from the category of A-modules to the category of $\mathscr{O}_{\operatorname{Spec} A}$-modules.

By applying the proposition to $M = A$, we also see that for an A-module N, $\widetilde{N}$ is zero if and only if N is zero.

The construction $M \mapsto \widetilde{M}$ is compatible with exactness, kernels, cokernels, images, direct sums, filtered inductive limits. (Cf. [GW1] Prop. 7.14 for a more precise statement.)

(2.4) Quasi-coherent modules.

Definition 2.11. *Let X be a ringed space. An $\mathscr{O}_X$-module $\mathscr{F}$ is called* quasi-coherent*, if every $x \in X$ has an open neighborhood U such that there exists an exact sequence*

$$\mathscr{O}_U^{(J)} \to \mathscr{O}_U^{(I)} \to \mathscr{F}_{|U} \to 0$$

for suitable (possibly infinite) index sets I, J.

For a morphism $f \colon X \to Y$ of ringed spaces and a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$, the pull-back $f^*\mathscr{G}$ is a quasi-coherent $\mathscr{O}_X$-module (since f^{-1} is exact and tensor product is a right exact functor). The direct image f_* preserves the property of quasi-coherence (only) under certain conditions.

Locally free $\mathscr{O}_X$-modules are quasi-coherent.

Clearly, for a ring A and an A-module M, $\widetilde{M}$ is a quasi-coherent $\mathscr{O}_{\operatorname{Spec} A}$-module. We will see below that the converse is true as well:

For a ringed space X and $f \in \Gamma(X, \mathscr{O}_X)$, we write $X_f := \{x \in X;\ f_x \in \mathscr{O}_{X,x}^\times\}$, an open subset of X. We obtain a homomorphism

$$\Gamma(X, \mathscr{F})_f \to \Gamma(X_f, \mathscr{F})$$

for every $\mathscr{O}_X$-module $\mathscr{F}$.

Theorem 2.12. *Let X be a scheme and $\mathscr{F}$ an $\mathscr{O}_X$-module. The following are equivalent:*

(i) *For every affine open* $\operatorname{Spec} A = U \subseteq X$*, there exists an A-module M such that $\mathscr{F}_{|U} \cong \widetilde{M}$.*
(ii) *There exists a covering $X = \bigcup_i U_i$ by affine open subschemes $U_i = \operatorname{Spec} A_i$ and A_i-modules M_i such that $\mathscr{F}_{|U_i} \cong \widetilde{M}_i$ for all i.*
(iii) *The $\mathscr{O}_X$-module $\mathscr{F}$ is quasi-coherent.*
(iv) *For every affine open* $\operatorname{Spec} A = U \subseteq X$ *and every $f \in A$, the homomorphism $\Gamma(U, \mathscr{F})_f \to \Gamma(D(f), \mathscr{F})$ is an isomorphism.*

Note that we can phrase (iv) equivalently as saying that the natural map $\Gamma(U, \mathscr{F})^\sim \to \mathscr{F}_{|U}$ is an isomorphism.

Sketch of proof. The implications (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are relatively easy. To show (iii) $\Rightarrow$ (iv), we may assume $X = U = \operatorname{Spec} A$ and we can cover X be finitely many principal open subsets $D(g_i)$ such that $\mathscr{F}_{|D(g_i)}$ is of the form $\widetilde{M}_i$. In particular, (iv) holds for $\mathscr{F}_{|D(g_i)}$, and similarly for $\mathscr{F}_{D(g_i g_j)}$. Now use the sheaf property of $\mathscr{F}$ to conclude that (iv) holds for U itself. □

Corollary 2.13. *Let A be a ring, $X = \operatorname{Spec} A$. The functor $\widetilde{\cdot}$ induces an exact equivalence between the categories of A-modules and of quasi-coherent $\mathscr{O}_X$-modules.*

The statements of the following corollary can be checked locally on X, hence it is enough to show the corresponding claims for modules in the image of the $\widetilde{\cdot}$ functor. For Part (3) use that tensor product is compatible with localization.

Corollary 2.14. *Let X be a scheme.*

(1) *Kernels, cokernels, images of $\mathscr{O}_X$-module homomorphisms between quasi-coherent $\mathscr{O}_X$-modules are quasi-coherent.*
(2) *Direct sums of quasi-coherent $\mathscr{O}_X$-modules are quasi-coherent.*
(3) *Let $\mathscr{F}$, $\mathscr{G}$ be quasi-coherent $\mathscr{O}_X$-module. Then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is quasi-coherent, and for every affine open $U \subseteq X$ we have*

$$\Gamma(U, \mathscr{F} \otimes \mathscr{G}) = \Gamma(U, \mathscr{F}) \otimes \Gamma(U, \mathscr{G}).$$

In particular, by (1) and (2) the category of quasi-coherent $\mathscr{O}_X$-module is an abelian category, and the inclusion functor into the category of all $\mathscr{O}_X$-modules preserves kernels and cokernels and direct sums.

(2.5) Direct and inverse image of quasi-coherent $\mathscr{O}_X$-module.

Proposition 2.15. *Let $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ be affine schemes, and let $f\colon X \to Y$ be a scheme morphism.*

1. *Let N be an B-module, then $f_*(\widetilde{N}) = \widetilde{N_{[A]}}$ where $N_{[A]}$ is N, considered as an A-module via $\Gamma(f)\colon A \to B$.*
2. *Let M be an A-module, then $f^*(\widetilde{M}) = \widetilde{M \otimes_A B}$.*

Sketch of proof. The first part can easily be checked directly. For the second part, use that we already know that $f^*\widetilde{M}$ is quasi-coherent, the Yoneda lemma and adjunction (or in other words, uniqueness of the left adjoint functor of f_*). □

(2.6) Finiteness conditions.

Definition 2.16. *Let X be a ringed space. We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is* of finite type *(or* of finite presentation*, resp.), if every $x \in X$ has an open neighborhood $U \subseteq X$ such that there exists $n \geq 0$ (or $m, n \geq 0$, resp.) and a short exact sequence*

$$\mathscr{O}_X^n \to \mathscr{F} \to 0$$

(or

$$\mathscr{O}_X^m \to \mathscr{O}_X^n \to \mathscr{F} \to 0,$$

resp.).

On an affine scheme, this coincides with the corresponding definitions in terms of modules (via $M \mapsto \widetilde{M}$). Note that every $\mathscr{O}_X$-module of finite presentation is quasi-coherent. On a noetherian scheme, every quasi-coherent $\mathscr{O}_X$-module of finite type is of finite presentation.

Proposition 2.17. *Let X be a ringed space and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite presentation.*

1. *For all $x \in X$ and for each $\mathscr{O}_X$-module $\mathscr{G}$, the canonical homomorphism of $\mathscr{O}_{X,x}$-modules*
$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \to \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$
is bijective.
2. *Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules of finite presentation. Let $x \in X$ be a point and let $\theta\colon \mathscr{F}_x \xrightarrow{\sim} \mathscr{G}_x$ be an isomorphism of $\mathscr{O}_{X,x}$-modules. Then there exists an open neighborhood U of x and an isomorphism $u\colon \mathscr{F}_{|U} \xrightarrow{\sim} \mathscr{G}_{|U}$ of $\mathscr{O}_U$-modules with $u_x = \theta$.*

Proof. Problem 15. □

Proposition 2.18. *Let X be a ringed space, and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type. Then the support*

$$\mathrm{Supp}(\mathscr{F}) = \{x \in X;\ \mathscr{F}_x \neq 0\}$$

of $\mathscr{F}$ is a closed subset of X.

Proof. Problem 18. □

(2.7) Closed subschemes and quasi-coherent ideal sheaves.

Proposition 2.19. *Let X be a scheme. An ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_X$ defines a closed subscheme if and only if $\mathscr{I}$ is a quasi-coherent $\mathscr{O}_X$-module.*

The point here is that both properties can be checked locally on X, and that for affine schemes we have already shown this statement (which amounts to saying that every closed subscheme of an affine scheme $\mathrm{Spec}\, A$ has the form $\mathrm{Spec}\, A/\mathfrak{a}$ for some ideal $\mathfrak{a} \subseteq A$.

We hence obtain an inclusion-reversing bijection between the set of closed subschemes of a scheme X and the set of quasi-coherent ideal sheaves in $\mathscr{O}_X$, mapping

- a quasi-coherent ideal sheaf $\mathscr{I}$ to $Z := (\mathrm{Supp}(\mathscr{O}_X/\mathscr{I}), i^{-1}(\mathscr{O}_X/\mathscr{I}))$, where $i\colon \mathrm{Supp}(\mathscr{O}_X/\mathscr{I}) \to X$ denotes the inclusion,
- a closed subscheme $Z \subseteq X$ to $\mathrm{Ker}(\mathscr{O}_X \to i_*\mathscr{O}_Z)$, where $i\colon Z \to X$ denotes the inclusion morphism.

We denote the closed subscheme corresponding to a quasi-coherent ideal sheaf $\mathscr{I}$ by $V(\mathscr{I})$.

(2.8) Locally free sheaves on affine schemes.

There is an obvious "commutative algebra way" of writing down, for an A-module M, the condition that $\widetilde{M}$ is locally free.

Theorem 2.20. *Let A be a ring and M an A-module. Consider the following properties of M:*

(i) *$\widetilde{M}$ is a locally free $\mathscr{O}_{\mathrm{Spec}\, A}$-module.*
(ii) *M is locally free, i.e., there exist $f_1, \dots, f_n \in A$ generating the unit ideal such that for all i, the A_{f_i}-module M_{f_i} is free.*
(iii) *For all $\mathfrak{p} \in \mathrm{Spec}\, A$, the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is free.*
(iv) *The A-module M is flat.*
(1) *We have the implications* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).
(2) *If M is an A-module of finite presentation, then all the four properties are equivalent.*

Proof. Part (1) is easy. Part (2) is more difficult. The implication (iii) $\Rightarrow$ (ii), for finitely presented M, follows from Prop. 2.17. See [GW1] Prop. 7.40.

There are several ways to show (iv) $\Rightarrow$ (iii). One can proceed in a fairly elementary fashion, using the "equational criterion of flatness", see [M2] Theorem 7.10, Theorem 7.12. Alternatively, using the Tor functor (the "left derived functor of the tensor product"), one can proceed as follows. We may assume that A is local with maximal ideal $\mathfrak{m}$. Lift the elements of an $A/\mathfrak{m}$-basis of $M/\mathfrak{m}M$ to M. By the Lemma of Nakayama this induces a surjection $A^n \to M$. Let K denote its kernel. We want to show that $K = 0$. Since M is of finite presentation, K is a finitely generated A-module ([M2] Theorem 2.6). Furthermore, the flatness of M implies that $\mathrm{Tor}_1^A(A/\mathfrak{m}, M) = 0$. Therefore the short exact sequence

$$0 \longrightarrow K \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

remains exact after tensoring $- \otimes_A A/\mathfrak{m}$. Since by construction the homomorphism $A^n \to M$ becomes an isomorphism after tensoring with the residue class field, this shows that $K \otimes_A A/\mathfrak{m}$. Applying the Lemma of Nakayama again, we obtain that $K = 0$, as desired. □

There is an obvious analogous theorem for $\mathscr{O}_X$-module on a scheme X, where we define

Definition 2.21. *Let X be a scheme. An $\mathscr{O}_X$-module $\mathscr{F}$ is called* flat*, if for all $x \in X$ the stalk $\mathscr{F}_x$ is a flat $\mathscr{O}_{X,x}$-module.*

More generally, given an $\mathscr{O}_X$-module $\mathscr{F}$ and a morphism $f\colon X \to Y$ we say that $\mathscr{F}$ is f-flat or flat over Y, if for all $x \in X$ the stalk $\mathscr{F}_x$ is a flat $\mathscr{O}_{Y,f(x)}$-module (via $f_x^\sharp\colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$).

If A is a domain, then every flat A-module M is torsion-free (i.e., multiplication by s is injective for all $s \in A \setminus \{0\}$). The converse holds only rarely; it does hold if A is a principal ideal domain and M is finitely generated.

Remark 2.22.

(1) Let A be a principal ideal domain. Then every finitely generated locally free (in the sense of condition (i′) in the theorem) A-module is free. (Use the structure theorem for finitely generated modules over principal ideal domains.)

(2) It is a difficult theorem (conjectured by Serre, proved independently by Quillen and Suslin) that every locally free sheaf of finite type on $\mathbb{A}^n_k$, k a field, is free. The same statement holds even for k a discrete valuation ring.

(3) It will not be relevant in the course, but in fact in the previous two items the hypothesis *of finite type* can be omitted. In fact, whenever R is a ring which is noetherian and such that $\operatorname{Spec} R$ is connected, then every locally free R-module which is *not finitely generated* is free. One way to show this is to combine the paper [Ba] by H. Bass with the difficult theorem that the property of a module of being "projective" can be

checked Zariski-locally on Spec A ([Stacks] `058B`), which shows that all locally free R-modules, finitely generated or not, are projective. Maybe there is also a more direct way, without talking about projective modules?

(4) On the other hand, even for an affine scheme X, a locally free $\mathscr{O}_X$-module is usually not a projective object in the category ($\mathscr{O}_X$-Mod).

(5) Let A be a noetherian unique factorization domain. Then every invertible sheaf on Spec A is free.

(6) See the answers to this question (`mathoverflow.net/q/54356`[1]) for examples of non-free locally free modules over Spec A for factorial (and even, in addition, regular) noetherian rings A.

(7) Let A be a domain, and let M be a locally free A-module of rank 1. Then M is isomorphic to a *fractional ideal*, i.e., to a finitely generated sub-A-module of $K := \mathrm{Frac}(A)$. (Cf. Problem 8 for a converse statement in the case that A is a Dedekind domain.)

[1] `https://mathoverflow.net/q/54356`

3. Line bundles and divisors

General references: [GW1] Ch. 11, in particular (11.9), (11.13); [H] II.6.

A *divisor* on a scheme X should be thought of an object that encodes a "configuration of zeros and poles (with multiplicities)" that a function on X could have. Below, we will see two ways to make this precise and compare them.

Let X be an integral (i.e., reduced and irreducible) scheme. We denote by $K(X)$ the field of rational functions of X.

Later we will impose the additional condition that X is noetherian and that all local rings $\mathscr{O}_{X,x}$ are unique factorization domains.

An important example that is good to keep in mind is the case of a *Dedekind scheme of dimension* 1, i.e., X is a noetherian integral scheme such that all points except for the generic point are closed, and such that for every closed point $x \in X$ the local ring $\mathscr{O}_{X,x}$ is a principal ideal domain (in other words: all local rings are discrete valuation rings), and the generic point is not closed itself. If a Dedekind scheme X is a k-scheme of finite type for some algebraically closed (or at least perfect) field k, then we call X a smooth algebraic curve over k.

Cartier divisors.

(3.1) Cartier divisors: Definition.

Denote by $K(X) = \mathscr{O}_{X,\eta}$ the field of rational functions on the integral scheme X, where $\eta \in X$ is the generic point. We denote by $\mathscr{K}_X$ the constant sheaf with value $K(X)$, i.e., $\mathscr{K}_X(U) = K(X)$ for all $\emptyset \neq U \subseteq X$ open. Since X is irreducible, this is a sheaf.

The notion of Cartier divisor encodes a zero/pole configuration by specifying, *locally on* X, functions with the desired zeros and poles. Since functions which are units in $\Gamma(U, \mathscr{O}_X)$ should be regarded as having no zeros and/or poles on U, we consider functions only up to units.

Definition 3.1. *A* Cartier divisor *on X is given by a tuple $(U_i, f_i)_i$, where $X = \bigcup_i U_i$ is an open cover, $f_i \in K(X)^\times$, and $f_i/f_j \in \Gamma(U_i \cap U_J, \mathscr{O}_X)^\times$ for all i, j. Two such tuples $(U_i, f_i)_i$, $(Vj, g_j)_j$ give rise to the same divisor, if $f_i g_j^{-1} \in \Gamma(U_i \cap V_j, \mathscr{O}_X)^\times$ for all i, j.*

With addition given by

$$(U_i, f_i)_i + (V_j, g_j)_j = (U_i \cap V_j, f_i g_j)_{i,j}$$

the set $\mathrm{Div}(X)$ of all Cartier divisors on X is an abelian group.

Remark 3.2. We have $\mathrm{Div}(X) = \Gamma(X, \mathscr{K}_X^\times / \mathscr{O}_X^\times)$.

Definition 3.3. *A Cartier divisor of the form* (X, f), $f \in K(X)^\times$, *is called a* principal *divisor. Divisors* D, D' *on* X *are called* linearly equivalent*, if* $D - D'$ *is a principal divisor. The set of principal divisors is a subgroup of* $\mathrm{Div}(X)$ *and the quotient* $\mathrm{DivCl}(X)$ *of* $\mathrm{Div}(X)$ *by this subgroup is called the* divisor class group *of* X.

(3.2) The line bundle attached to a Cartier divisor.

Let D be a Cartier divisor on X. We define an invertible $\mathscr{O}_X$-module $\mathscr{O}_X(D)$ as follows:

$$\Gamma(U, \mathscr{O}_X(D)) = \{f \in K(X); \forall i : f_i f \in \Gamma(U \cap U_i, \mathscr{O}_X)\} \quad \text{for } \emptyset \neq U \subseteq X \text{ open.}$$

For each i, we have $\mathscr{O}_X(D)_{|U_i} = f_i^{-1}\mathscr{O}_{U_i} \subset \mathscr{K}_X$, so multiplication by f_i gives an $\mathscr{O}_{U_i}$-module isomorphism $\mathscr{O}_X(D)_{|U_i} \cong \mathscr{O}_{U_i}$.

Proposition 3.4. *The map* $D \mapsto \mathscr{O}_D(X)$ *induces group isomorphisms* $\mathrm{Div}(X) \cong \{\mathscr{L} \subset \mathscr{K}_X$ *invertible* $\mathscr{O}_X$*-module*$\}$ *and* $\mathrm{DivCl}(X) \cong \mathrm{Pic}(X)$.

Sketch of proof. To construct an inverse of the map $D \mapsto \mathscr{O}_D(X)$, take $\mathscr{L} \subseteq \mathscr{K}_X$ invertible and choose an open cover $X = \bigcup U_i$ such that $\mathscr{L}_{|U_i}$ is trivial for each i. Then necessarily $\mathscr{L}_{|U_i} = f_i^{-1}\mathscr{O}_{U_i}$ for some $f_i \in K(X)^\times$ (namely, f_i^{-1} is the image of $1 \in \Gamma(U_i, \mathscr{O}_X)$ under the map $\Gamma(U_i, \mathscr{O}_X) \to K(X)$ induced by the composition $\mathscr{O}_{U_i} \cong \mathscr{L}_{|U_i} \to \mathscr{K}_X$). We then map $\mathscr{L}$ to the Cartier divisor $(U_i, f_i)_i$. One checks that this map is well-defined (i.e., independent of the choice of cover and of the choice of the elements f_i) and that the two maps are inverse to each other.

It remains to check that $\mathscr{O}_D(X)$ is free if and only if D is principal (Problem 28) and that every invertible $\mathscr{O}_X$-module $\mathscr{L}$ can be embedded as a submodule into $\mathscr{K}_X$. This is easy if X is affine. In the general case, let $U \subseteq X$ be open affine. We claim that every embedding $\mathscr{L}_{|U} \hookrightarrow \mathscr{K}_U$ extends uniquely to an embedding $\mathscr{L} \hookrightarrow \mathscr{K}_X$. Because of the uniqueness, we can work locally on X (and afterwards use gluing of sheaf homomorphisms), and therefore restrict to the case $\mathscr{L} = \mathscr{O}_X$. The embedding $\mathscr{O}_U = \mathscr{L}_{|U} \hookrightarrow \mathscr{K}_U$ then corresponds to a section $s \in \Gamma(U, \mathscr{K}_U)^\times$. But $\Gamma(U, \mathscr{K}_U) = K(X) = \Gamma(X, \mathscr{K}_X)$, so the claim follows. (See [GW1] Prop. 11.29 for more details and a variant which does not require X to be integral.) □

To get a more geometric view on divisors, a first step is the following definition of the support of a divisor. We will carry this further by introducing the notion of Weil divisor, see below, and relating it to Cartier divisors.

Definition 3.5. *The* support *of a Cartier divisor* D *is*

$$\mathrm{Supp}(D) = \{x \in X;\ f_{i,x} \in K(X)^\times \setminus \mathscr{O}_{X,x}^\times \text{ (where } x \in U_i)\},$$

a proper closed subset of X.

Weil divisors.

Now let X be a noetherian integral scheme, such that all local rings $\mathscr{O}_{X,x}$ are unique factorization domains. (The theory can be set up in more generality, see [GW1] Section (11.13).) This is true for example if all local rings of X are regular local rings.

(Recall that a noetherian local ring A is called regular, if its maximal ideal can be generated by $\dim A$ elements, where $\dim A$ is the Krull dimension of A. The Krull dimension is the supremum of the lengths of inclusion chains of prime ideals in A. For instance, a local ring A has dimension 0 if and only if its maximal ideal is also a minimal prime ideal (and hence the unique prime ideal of A). A local ring A has dimension 1 if it maximal ideal is not a minimal prime ideal, but all non-maximal prime ideals are minimal prime ideals. A local domain has dimension 1 if and only if it has precisely two prime ideals.)

(3.3) Definition of Weil divisors.

Let $Z^1(X)$ denote the free abelian group on maximal proper integral subschemes of X (equivalently: those integral subschemes $Z \subset X$ such that for the generic point $\eta_Z \in Z$ we have $\dim \mathscr{O}_{X,\eta_Z} = 1$). We say that Z *has codimension* 1. We also write $\mathscr{O}_{X,Z} := \mathscr{O}_{X,\eta_Z}$.

By our assumptions on X, all the rings $\mathscr{O}_{X,Z}$ are discrete valuation rings. (Since they are noetherian domains of dimension 1 by assumption, it is equivalent to require that they are integrally closed, or factorial, or that they are regular.) We denote by $v_Z\colon K(X)^\times \to \mathbb{Z}$ the corresponding discrete valuation on K, and set $v_Z(0) = \infty$.

Definition 3.6. *An element of $Z^1(X)$ is called a* Weil divisor*. We write Weil divisors as finite "formal sums" $\sum n_Z[Z]$ where $Z \subset X$ runs through the integral closed subschemes of X of codimension* 1*.*

For $f \in K(X)^\times$, we define the divisor attached to f as

$$\operatorname{div}(f) = \sum_Z v_Z(f)[Z].$$

Note that the sum is finite, i.e., $v_Z(f) = 0$ for all but finitely many Z. In fact, for $U \subseteq X$ affine open, the complement $X \setminus U$ has only finitely many irreducible components, so we may discard it and replace X by U. Then assume $X = \operatorname{Spec} A$ is affine and write $f = g/h$ with $g, h \in A$. Then $v_Z(f)$ can only be $\neq 0$, if Z is an irreducible component of $V(f) \cup V(g)$. Since this closed subscheme of the noetherian scheme X has only finitely many irreducible components (being itself noetherian), we are done.

Weil divisors of the form $\operatorname{div}(f)$ are called *principal Weil divisors*. Two Weil divisors are called *linearly equivalent*, if their difference is a principal divisor.

Definition 3.7.

(1) *A Weil divisor* $\sum_Z n_Z[Z]$ *is called* effective*, if* $n_Z \geq 0$ *for all* Z*.*

(2) *A Cartier divisor* D *is called* effective*, if* $\mathscr{O}_X \subseteq \mathscr{O}_X(D)$ *(inside* $\mathscr{K}_X$*), or equivalently, if* $\mathscr{O}_X(-D) \subseteq \mathscr{O}_X$ *is an ideal of* $\mathscr{O}_X$*.*

(3.4) Weil divisors vs. Cartier divisors.

Generalizing the definition of principal divisors, we can construct a group homomorphism $\mathrm{cyc}\colon \mathrm{Div}(X) \to Z^1(X)$ as follows:

$$D = (U_i, f_i) \mapsto \sum v_Z(f_{i_Z})[Z],$$

where for each Z we choose an index i_Z so that U_{i_Z} contains the generic point of Z (equivalently: $U_{i_Z} \cap Z \neq \emptyset$).

To prove that cyc is an isomorphism $\mathrm{Div}(X) \cong Z^1(X)$, we need the following facts from commutative algebra. (See, e.g., [M2] Theorem 11.5, Theorem 20.1.) Recall that a domain A is called *integrally closed*, or *normal*, if every element of the field of fractions of A which is the zero of a monic polynomial with coefficients in A lies in A. Every UFD is integrally closed. Furthermore, a domain A is integrally closed if and only if all localizations $A_\mathfrak{p}$ at prime ideals $\mathfrak{p} \in \operatorname{Spec} A$ are integrally closed. So for X as above, and a non-empty affine open $U \subseteq X$, the ring $\Gamma(U, \mathscr{O}_X)$ is integrally closed (in its field of fractions $K(X)$).

Lemma 3.8.

(1) *Let* A *be an integrally closed domain. Then* A *is equal to the intersection of all localizations* $A_\mathfrak{p}$ *(in* $\mathrm{Frac}(A)$*), where* $\mathfrak{p}$ *runs through the set of minimal prime ideals of* A*.*

(2) *Let* A *be a local unique factorization domain. Then every prime ideal* $\mathfrak{p} \neq 0$ *which is minimal among all prime ideals* $\neq 0$ *of* A *is a principal ideal.*

Proposition 3.9. *The map* cyc *is a group isomorphism* $\mathrm{Div}(X) \cong Z^1(X)$*. Under this isomorphism, the subgroups of principal divisors on each side correspond to each other, whence it induces an isomorphism* $\mathrm{DivCl}(X) \cong \mathrm{Cl}(X)$*.*

Sketch of proof. Injectivity. If D is a Weil divisor or a Cartier divisor such that D and $-D$ are effective, then D is trivial. It therefore suffices to show that the inverse image of the subset of effective Weil divisors under the homomorphism cyc consists of effective Cartier divisors. So let D be a Cartier divisor on X such that $\mathrm{cyc}(D)$ is effective. We can check that D is effective locally on X, so we may assume that $X = \operatorname{Spec} A$ for an integrally closed domain A, and that D is principal, say given by (X, f). By assumption $f \in K(X)$ is contained in $A_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$, and it follows from Lemma 3.8 Part (1) that $f \in A$, as desired.

Surjectivity. We construct an inverse to the map cyc. If $Z \subset X$ is an integral closed subscheme of X of codimension 1 with corresponding ideal sheaf $\mathscr{I}_Z \subseteq \mathscr{O}_X$, then for every $x \in X$, $\mathscr{I}_{Z,x}$ is a principal ideal in $\mathscr{O}_{X,x}$ by Lemma 3.8 Part (2). Using Proposition 2.17, we find an affine open cover $(U_i)_i$ of X together with elements $f_i \in K(X)$ such that $\mathscr{I}_{Z|U_i} = f_i\mathscr{O}_X$ (inside $\mathscr{K}_X$), for each i. We then map $[Z]$ to the Cartier divisor $(U_i, f_i)_i$ (well-defined since elements in a domain that generate the same principal ideal differ at most by a unit), and extend this construction to a map $d\colon Z^1(X) \to \operatorname{Div}(X)$ by linearity. By construction we have $\operatorname{cyc} \circ d = \operatorname{id}$, and this implies that cyc is surjective (and hence bijective with inverse d).

We can phrase the definition of principal Weil divisor as saying that it is the image under cyc of a principal Cartier divisor. It is therefore clear that we also obtain an isomorphism $\operatorname{DivCl}(X) \cong \operatorname{Cl}(X)$. □

We thus have identifications

$$\operatorname{Pic}(X) \cong \operatorname{DivCl}(X) \cong \operatorname{Cl}(X).$$

Example 3.10.

(1) For any UFD A, $\operatorname{Pic}(A) = 1$ as remarked above. In particular, all divisors on affine space $\mathbb{A}^n_k$ over a field (or over any UFD) k are principal.

(2) Let k be a field. As shown on the problem sheets, $\operatorname{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}$. We will see below that $\operatorname{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$ for every $n \geq 1$.

The Picard group or equivalently the divisor class group of an integral scheme X contains interesting information about X, but is often not easy to compute.

(3.5) The theorem of Riemann and Roch.

No proofs were given in the lecture at this point for the following results.
Reference: [H] IV.1.

Now let X be a Dedekind scheme which is a scheme of finite type over an algebraically closed field k. In view of Proposition 3.9, we identify Cartier and Weil divisors. In addition we assume that X is projective, i.e., that there exist $n \geq 1$ and a closed immersion $X \hookrightarrow \mathbb{P}^n_k$.

For a (Weil) divisor $D = \sum_Z n_Z[Z]$ we define the *degree* $\deg(D)$ of D as $\deg(D) := \sum_Z n_Z$. We obtain a group homomorphism $Z^1(X) \to \mathbb{Z}$. Under our assumption that X is a closed subscheme of some projective space, one can show that this homomorphism factors through $\operatorname{Cl}(X)$:

Theorem 3.11. *Let $f \in K(X)$. Then* $\deg(\operatorname{div}(f)) = 0$.

This means that the degree homomorphism $\operatorname{Div}(X) \to \mathbb{Z}$ factors through the divisor class group. In particular, we can speak of the degree of a line bundle, and we denote the degree of $\mathscr{L}$ by $\deg(\mathscr{L})$.

To state the famous Theorem of Riemann–Roch, we introduce the following notation. For a divisor D we write $\ell(D) = \dim_k \Gamma(X, \mathscr{O}_X(D))$.

Proposition 3.12. *For each D, $\ell(D)$ is finite.*

If $\ell(D) \geq 0$, then $\deg(D) \geq 0$. In fact, it was shown that whenever $\ell(D) \neq 0$, then D is linearly equivalent to an effective divisor D', and then $\deg(D) = \deg(D') \geq 0$.

Theorem 3.13. (Riemann-Roch) *For X as above, there exist $g \in \mathbb{Z}_{\geq 0}$ and $K \in \mathrm{Div}(X)$ such that for every divisor D on X, we have*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

Corollary 3.14. *In the above situation, we have*
(1) $\ell(K) = g$,
(2) $\deg(K) = 2g - 2$,
(3) *for every D with $\deg(D) > 2g - 2$, we have $\ell(D) = \deg(D) + 1 - g$.*

Proof. The corollary is easy to prove with the Theorem of Riemann-Roch at hand. In fact, for (1) use the theorem with $D = 0$ the trivial divisor, for (2) use $D = K$, and for (3) use that under the assumption there $\deg(K - D) < 0$, whence $\ell(K - D) = 0$, as remarked before. □

The number g is called the *genus* of the curve X. Part (3) of the corollary shows that it is uniquely determined by X.

Remark 3.15. The linear equivalence class of the canonical divisor K is uniquely determined. In fact, assume that K and K' are divisors which both have the property of a canonical divisor as in the Riemann-Roch theorem. Using the theorem and the corollary, one computes that $\ell(K - K') > 0$ and $\ell(K' - K) > 0$. As was shown on the problem sheet, this implies that $\mathscr{O}_X(K - K')$ is trivial, or in other words that K and K' have the same divisor class.

For the projective line $\mathbb{P}^1_k$, it is easy to prove the Theorem of Riemann-Roch by direct computations. It has genus 0. One can show that every X of genus 0 is isomorphic to $\mathbb{P}^1_k$. (But if k is not assumed to be algebraically closed, then there may exist X as above of genus 0 which are not isomorphic to $\mathbb{P}^1_k$.)

For X as above which is of the form $V_+(f) \subset \mathbb{P}^2_k$, there is the following formula for the genus:

Proposition 3.16. *Let X as above be of the form $V_+(f) \subset \mathbb{P}^2_k$ for a homogeneous polynomial f of degree d. Then the genus g of X is given by*

$$g = \frac{(d-1)(d-2)}{2}.$$

For example, elliptic curves (which are defined by a homogeneous polynomial of degree 3) have genus 1.

(3.6) Line bundles on $\mathbb{P}^n_k$.

References: [GW1], Ch. 8, Ch. 11, in particular Example 11.43, (8.5); [H] II.6, II.7.

We want to compute the Picard group of projective space over a field. To this end, we will use the following general proposition.

Proposition 3.17. *Let X be a noetherian integral scheme such that all local rings $\mathscr{O}_{X,x}$ are unique factorization domains. Let $U \subseteq X$ be an open subscheme, and let $Z_1, \dots, Z_r$ be those irreducible components of $X \setminus U$ that are of codimension 1 inside X. We consider the Z_i as integral closed subschemes of X. Then we have a short exact sequence*

$$0 \to \bigoplus_i \mathbb{Z}[Z_i] \to Z^1(X) \to Z^1(U) \to 0$$

which induces an exact sequence

$$\bigoplus_i \mathbb{Z}[Z_i] \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0.$$

Proof. It is clear that we have the first short exact sequence when we think of integral closed subschemes in terms of their generic points. It is easy to check that the "restriction map" $Z^1(X) \to Z^1(U)$ induces a homomorphism between the class groups, and this yields the second exact sequence. □

In terms of the identifications of the divisor class groups with the Picard groups of X and of U, the map $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ in the proposition is just the restriction of line bundles from X to the open subscheme U.

Now let R be a ring and fix $n \geq 1$. We cover $\mathbb{P}^n_R$ by the standard charts $U_i := D_+(X_i)$, as usual, and write $U_{ij} := U_i \cap U_j$. For $d \in \mathbb{Z}$, multiplication by the elements $(X_i/X_j)^d \in \Gamma(U_{ij}, \mathscr{O}_{\mathbb{P}^n_R})^\times$ defines isomorphisms $\mathscr{O}_{U_i|U_{ij}} \to \mathscr{O}_{U_j|U_{ij}}$ which give rise to a gluing datum of the $\mathscr{O}_{U_i}$-modules $\mathscr{O}_{U_i}$. By gluing of sheaves, we obtain a line bundle $\mathscr{O}_{\mathbb{P}^n_R}(d)$. (Cf. Problems 23, 24, 25 in the case $n = 1$.) To shorten the notation, we sometimes just write $\mathscr{O}(d)$, when the space is clear from the context.

Lemma 3.18. *We obtain a group homomorphism $\mathbb{Z} \to \mathrm{Pic}(\mathbb{P}^n_R)$, $d \mapsto \mathscr{O}(d)$.*

Proposition 3.19. *Writing $R[X_0, \dots, X_n]_d$ for the submodule of homogeneous polynomials of degree d (with $R[X_0, \dots, X_n]_d = 0$ for $d < 0$), we have natural isomorphisms*

$$\Gamma(\mathbb{P}^n_R, \mathscr{O}(d)) \cong R[X_0, \dots, X_n]_d$$

for all $d \in \mathbb{Z}$.

Proof. We can make the gluing construction for $\mathscr{O}(d)$ explicit by identifying $\Gamma(D_+(X_i), \mathscr{O}(d))$ with $X_i^d R\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$, and correspondingly

$$\Gamma(D_+(X_iX_j), \mathscr{O}(d)) = X_i^d R\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}, \frac{X_i}{X_j}\right] = X_j^d R\left[\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}, \frac{X_j}{X_i}\right]$$

(inside $R[X_0, \dots, X_n, X_0^{-1}, \dots, X_n^{-1}]$). This means that the restriction map is just the inclusion map. (If R is a domain, then this describes an embedding of $\mathscr{O}(d)$ into the constant sheaf $\mathscr{K}_{\mathbb{P}^n_R}$ with values the field of rational functions.)

With this description, since the relevant restriction maps are injective, we can identify $\Gamma(\mathbb{P}^n_R, \mathscr{O}(d))$ with the intersection

$$\bigcap_{i=0}^{n} X_i^d R\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right].$$

One checks that this intersection is $R[X_0, \dots, X_n]_d$, as claimed. □

Corollary 3.20. *The above homomorphism $\mathbb{Z} \to \operatorname{Pic}(\mathbb{P}^n_R)$, $d \mapsto \mathscr{O}(d)$, is injective.*

Now let $R = k$ be a field (in fact, the same arguments apply to any noetherian unique factorization domain k). Then $\mathbb{P}^n_k$ is a noetherian integral scheme all of whose local rings are unique factorization domains, so we can talk about Cartier divisors and about Weil divisors, and identify the two notions via the cycle map as in Proposition 3.9.

Corollary 3.21. *Let k be a field and let $n \geq 1$. Then $\operatorname{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$.*

Proof. If we identify $\operatorname{Cl}(\mathbb{P}^n_k) = \operatorname{Pic}(\mathbb{P}^n_k)$ and apply Proposition 3.17 to $X = \mathbb{P}^n_k$ and $U = D_+(X_0) \cong \mathbb{A}^n_k$, we obtain a surjection $\mathbb{Z} \to \operatorname{Pic}(\mathbb{P}^n_k)$. The injectivity statement of the previous corollary implies that this surjection (which allos us to identify $\operatorname{Pic}(\mathbb{P}^n_k)$ with some quotient of $\mathbb{Z}$) must be an isomorphism. (At this point we have not yet shown that the map $d \mapsto \mathscr{O}(d)$ is surjective, and hence an isomorphism; this will follow from the discussion in the following section.) □

Remark 3.22. One can show that every locally free $\mathscr{O}_{\mathbb{P}^1_k}$-module is isomorphic to a direct sum of line bundles (Problem 27). Note though that this statement is not true for $\mathbb{P}^n_k$, $n > 1$.

(3.7) Divisors on $\mathbb{P}^n_k$.

Let k be a field (or more generally a noetherian unique factorization domain), and let $n \geq 1$. Let us take a look at the line bundles $\mathscr{O}(d)$ from the point of view of Cartier divisors. Write

$$\mathcal{R} = \{f = \frac{g}{h};\ g, h \in k[X_0, \dots, X_n] \text{ non-zero homogeneous polynomials}\}.$$

For $f = g/h \in \mathcal{R}$, we define $\deg(f) = \deg(g) - \deg(h)$. We can identify $K(X)^\times$ with the subgroup of $\mathcal{R}$ of degree 0 elements.

Fix an element $f \in \mathcal{R}$ and write $d = \deg(f)$. Let D be the Cartier divisor $\operatorname{div}(f) := (D_+(X_i), f/X_i^d)_i$ (this is a new use of the symbol div since f is not an element of $K(\mathbb{P}^n_k)$). Describing the line bundle $\mathscr{O}_{\mathbb{P}^n_k}(D)$ in terms of a gluing datum, it follows that $\mathscr{O}_X(D) \cong \mathscr{O}(d)$. Thus the composition

$$\mathcal{R} \to \operatorname{Div}(\mathbb{P}^n_k) \to \operatorname{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$$

is the degree map on $\mathcal{R}$. In particular, the isomorphism class of the line bundle $\mathscr{O}_{\mathbb{P}^n_k}(\operatorname{div}(f))$ depends only on d, not on the choice of f.

Now let $f \in k[X_0, \dots, X_n]$ be an irreducible homogeneous polynomial of degree $d > 0$. Then $V_+(f)$ is an integral closed subscheme of $\mathbb{P}^n_k$ of codimension 1 (since the same is true after intersection with any of the open charts $D_+(X_i)$ (unless the intersection is empty)). From the construction of the matching between Cartier and Weil divisors, one sees that the Cartier divisor $\operatorname{div}(f)$ defined above corresponds to the Weil divisor $[V_+(f)]$. As a particular example, for any fixed i, the Weil divisor $[V_+(X_i)]$ of the line $V_+(X_i)$ has associated line bundle $\mathscr{O}(1)$.

Since the identification of Cartier divisors with Weil divisors is a group isomorphism, one can extend this description to all divisors, by decomposing a general $f \in \mathcal{R}$ as a product of irreducible homogeneous polynomials and of inverses of such polynomials.

Coming back to the case of an irreducible homogeneous polynomial f of degree $d > 0$, the datum of the divisor $\operatorname{div}(f)$ corresponds to the choice of embedding of its associated line bundle $\mathscr{O}(d)$ into $\mathscr{K}_{\mathbb{P}^n_k}$. The image of this embedding contains the structure sheaf $\mathscr{O}_{\mathbb{P}^n_k}$, and going through the definitions shows that the global section $1 \in \Gamma(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k})$ is mapped to $f \in k[X_0, \dots, X_n]_d = \Gamma(\mathbb{P}^n_k, \mathscr{O}(d))$ (Proposition 3.19) under this embedding. Compare Problems 31, 33. In other words, the embedding $\Gamma(\mathbb{P}^n_k, \mathscr{O}(d)) \to K(X)$ of the global sections is given by

$$k[X_0, \dots, X_n]_d \to K(X), \quad g \mapsto g/f.$$

Corollary 3.23. *Let k be a field, and let $Z \subseteq \mathbb{P}^n_k$ be an integral closed subscheme of codimension 1. Then $Z = V_+(f)$ for some homogeneous polynomial f.*

Proof. Consider the divisor $[Z]$ given by Z. Viewed as a Cartier divisor, it corresponds to an embedding $\mathscr{O}_{\mathbb{P}^n_k}([Z]) \to \mathscr{K}_{\mathbb{P}^n_k}$ whose image contains $\mathscr{O}_{\mathbb{P}^n_k}$ since the divisor $[Z]$ is effective. Let $f \in k[X_0, \dots, X_n]_d$ be the image of $1 \in \Gamma(\mathbb{P}^n_k, \mathscr{O})$ in $\Gamma(\mathbb{P}^n_k, \mathscr{O}([Z])) = k[X_0, \dots, X_n]_d$ under this embedding. Since the embedding $\mathscr{O}_{\mathbb{P}^n_k}([Z]) \to \mathscr{K}_{\mathbb{P}^n_k}$ is entirely determined by this image, the above discussion shows that $[Z] = [V_+(f)]$ as divisors and hence that $Z = V_+(f)$. □

(3.8) Functorial description of $\mathbb{P}^n$.

As we have seen in Section1.1, every scheme X defines a contravariant functor $T \mapsto X(T) := \operatorname{Hom}_{(\mathrm{Sch})}(T, X)$ from the category of schemes to the category of sets. This functor determines X up to unique isomorphism. In this section, we want to describe the functor attached in this way to projective space $\mathbb{P}^n_R$ for R a ring.

Lemma 3.24. *Let X be a scheme.*

(1) *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Giving an $\mathscr{O}_X$-module homomorphism $\alpha\colon {}^{n+1}_X \to \mathscr{F}$ is "the same" as giving global sections $s_0, \dots, s_n \in \Gamma(X, \mathscr{F})$ (namely the images of the standard basis vectors of $\Gamma(X, \mathscr{O}_X^{n+1}) = \Gamma(X, \mathscr{O}_X)^{n+1}$.*
(2) *Now let $\mathscr{L}$ be a line bundle on X, and let $\alpha\colon \mathscr{O}_X^{n+1} \to \mathscr{L}$ be an $\mathscr{O}_X$-module homomorphism given by $s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$. Then α is surjective if and only if for every $x \in X$ there exists i such that $s_i(x) \neq 0$ in the fiber $\mathscr{L}(x)$.*

Proposition 3.25. *Let R be a ring, and let S be an R-scheme. There are bijections, functorial in S,*

$$\mathbb{P}^n_R(S) = \{(\mathscr{L}, \alpha);\ \mathscr{L} \text{ a line bundle on } S,$$
$$\alpha\colon \mathscr{O}_S^{n+1} \twoheadrightarrow \mathscr{L} \text{ a surjective } \mathscr{O}_S\text{-module homom.}\}/\cong .$$

Here we consider pairs $(\mathscr{L}, \alpha)$, $(\mathscr{L}', \alpha')$ as isomorphic, if there exists an $\mathscr{O}_S$-module isomorphism $\beta\colon \mathscr{L} \to \mathscr{L}'$ with $\alpha = \alpha' \circ \beta$.

Saying that the bijections of the proposition are functorial means that given a morphism $S' \to S$ of R-schemes the bijections for S and S' together with the natural map $\mathbb{P}^n_R(S) \to \mathbb{P}^n_R(S')$ and the map $(\mathscr{L}, \alpha) \mapsto (g^*\mathscr{L}, g^*\alpha)$ give rise to a commutative diagram. (Note that for every line bundle $\mathscr{L}$ on S the pull-back $g^*\mathscr{L}$ is a line bundle on S', and for surjective α the pull-back $g^*\alpha$ is again a surjective $\mathscr{O}_{S'}$-homomorphism of the desired form.)

Proof. A homomorphism $\alpha\colon \mathscr{O}_S^{n+1} \twoheadrightarrow \mathscr{L}$ corresponds to $n+1$ global sections in $\Gamma(S, \mathscr{L})$ (the "images of the standard basis vectors"). Thus $X_0, \dots, X_n \in \Gamma(\mathbb{P}^n_R, \mathscr{O}(1))$ give rise to a homomorphism $\mathscr{O}_{\mathbb{P}^n_R}^{n+1} \to \mathscr{O}(1)$. This homomorphism is surjective. (In fact, looking back at the construction of $\mathscr{O}(1)$ by gluing and the way how we identified the global sections of $\mathscr{O}(1)$ with $R[X_0, \dots, X_n]_1$, under the identification $\mathscr{O}(1)_{D_+(X_i)} \cong \mathscr{O}_{D_+(X_i)}$ the restriction of X_i to $D_+(X_i)$ corresponds to $1 \in \Gamma(D_+(X_i), \mathscr{O}_{D_+(X_i)})$ and in particular is non-zero in every fiber.)

Given a morphism $S \to \mathbb{P}^n_R$, we can pull this homomorphism back to S and obtain an element of the right hand side in the statement of the proposition.

Conversely, given a pair $(\mathscr{L}, \alpha)$ on S, we can think of the corresponding morphism $S \to \mathbb{P}^n_R$ in terms of homogeneous coordinates (i.e., for K-valued points for some field K), as follows: Denote by $f_0, \dots, f_n \in \Gamma(S, \mathscr{L})$ the global sections corresponding to α. For a point $x \in S$, the fiber $\mathscr{L}(x)$ is a one-dimensional $\kappa(x)$-vector space generated by the elements $f_0(x), \dots, f_n(x)$ (i.e., at least one of them is $\neq 0$ – this holds since α is surjective). We choose an isomorphism $\mathscr{L}(x) \cong \kappa(x)$, and hence can view the $f_i(x)$ as elements of $\kappa(x)$. Then the morphism $S \to \mathbb{P}^n_S$ maps x to $(f_0(x) : \dots : f_n(x)) \in \mathbb{P}^n(\kappa(x))$. While the individual $f_i(x)$, as elements of $\kappa(x)$, depend on the choice of isomorphism $\mathscr{L}(x) \cong \kappa(x)$, the point $(f_0(x) : \dots : f_n(x)) \in \mathbb{P}^n(\kappa(x))$ is independent of this choice.

To make this rigorous, consider a pair $(\mathscr{L}, \alpha)$ as above, and for $i \in \{0, \dots, n\}$ define

$$S_i = \{s \in S;\ \alpha(e_i) \text{ generates the fiber } \mathscr{L}(s)\},$$

where $e_i \in \mathscr{O}_S^{n+1}$ denotes the i-th standard basis vector. This defines an open cover of S. By definition, composing α with the injection $\mathscr{O}_S \to \mathscr{O}_S^{n+1}$ as the i-th summand induces a trivialization $\mathscr{O}_{S_i} \cong \mathscr{L}_{|S_i}$ of the restriction of $\mathscr{L}$. We obtain a morphism $S_i \to D_+(X_i) = \operatorname{Spec} k[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}]$ by mapping $\frac{X_j}{X_i}$ to the image of $\alpha(e_j) \in \Gamma(S_i, \mathscr{L})$ under the isomorphism $\Gamma(S_i, \mathscr{L}) \to \Gamma(S_i, \mathscr{O}_S)$. These morphisms can be glued, and one obtains the desired morphism $S \to \mathbb{P}^n_R$.

To conclude the proof, one checks that the two constructions are inverse to each other. □

Remark 3.26. If $S = \operatorname{Spec} k$ for a field k, then every locally free sheaf on S is free, and the proposition reads as

$$\mathbb{P}^n(k) = \{\alpha\colon k^{n+1} \to k \text{ surjective}\}/\cong,$$

where now homomorphisms α, α' are isomorphic if and only if they have the same kernel. Thus we can identify

$$\mathbb{P}^n(k) = \{U \subset k^{n+1} \text{ sub-vector space of dimension } n\}.$$

This description is dual to the classical description of $\mathbb{P}^n(k)$ as the set of lines in k^{n+1}. Passing to the dual space, the projection $k^{n+1} \to k^{n+1}/U$ induces an inclusion $(k^{n+1}/U)^\vee \to k^{n+1,\vee}$ of the dual vector spaces. Matching the standard basis of k^{n+1} with its dual basis, we can identify $k^{n+1,\vee} = k^{n+1}$, and in this way we get back the description in terms of lines.

4. Smoothness and differentials *

General reference: [GW1] Ch. 6.

The Zariski tangent space.

(4.1) Definition of the Zariski tangent space.

Definition 4.1. *Let X be a scheme, $x \in X$, $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ the maximal ideal in the local ring at x, $\kappa(x)$ the residue class field of X in x. The $\kappa(x)$-vector space $(\mathfrak{m}/\mathfrak{m}^2)^*$ is called the* (Zariski) tangent space of X in x.

Definition 4.2. *Let R be a ring, $f_1, \dots, f_r \in R[T_1, \dots, T_n]$. We call the matrix*

$$J_{f_1,\dots,f_r} := \left(\frac{\partial f_i}{\partial T_j}\right)_{i,j} \in M_{r\times n}(R[T_\bullet])$$

the Jacobian matrix *of the polynomials f_i. Here the partial derivatives are to be understood in a formal sense.*

Remark 4.3.

(1) If in the above setting the ideal $\mathfrak{m}$ is finitely generated, then $\dim_{\kappa(x)} T_x X$ is the minial number of elements needed to generate $\mathfrak{m}$ and in particular is finite.
(2) The tangent space construction if functorial in the following sense: Given a scheme morphism $f\colon X \to Y$ and $x \in X$ such that $\dim_{\kappa(x)} T_x X$ is finite or $[\kappa(x) : \kappa(f(x))]$ is finite, then we obtain a map

$$df_x\colon T_x X \to T_{f(x)} Y \otimes_{\kappa(f(x))} \kappa(x).$$

Example 4.4. Let k be a field, $X = V(f_1, \dots, f_m) \subseteq \mathbb{A}^n_k$, $f_i \in k[T_1, \dots, T_n]$, $x = (x_i)_i \in k^n = \mathbb{A}^n(k)$. Then there is a natural identification $T_x X = \operatorname{Ker}(J_{f_1,\dots,f_m}(x))$, where $J_{f_1,\dots,f_m}(x)$ denotes the matrix with entries in $\kappa(x) = k$ obtained by mapping each entry of $J_{f_1,\dots,f_m}$ to $\kappa(x)$, which amounts to evaluating these polynomials at x.

Proposition 4.5. *Let k be a field, X a k-scheme, $x \in X(k)$. There is an identification (functorial in (X, x))*

$$X(k[\varepsilon]/(\varepsilon^2))_x := \{f \in \operatorname{Hom}_k(\operatorname{Spec} k[\varepsilon]/(\varepsilon^2), X);\ \operatorname{im}(f) = \{x\}\} = T_x X.$$

Smooth morphisms.

(4.2) Definition of smooth morphisms.

Definition 4.6. *A morphism $f\colon X \to Y$ of schemes is called* smooth of relative dimension $d \geq 0$ in $x \in X$*, if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \operatorname{Spec} R \subseteq Y$ of $f(x)$ such that $f(U) \subseteq V$ and an open immersion $j\colon U \to \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d})$ such that the triangle*

$$\begin{array}{ccc} U & \xrightarrow{\ j\ } & \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d}) \\ & {\scriptstyle f}\searrow & \swarrow \\ & V & \end{array}$$

is commutative, and that the Jacobian matrix $J_{f_1,\dots,f_{n-d}}(x)$ has rank $n-d$.

We say that $f\colon X \to Y$ is smooth of relative dimension d *if f is smooth of relative dimension d at every point of X. Instead of* smooth of relative dimension 0*, we also use the term* étale*.*

With notation as above, if f is smooth at $x \in X$, then x as an open neighborhood such that f is smooth at all points of this open neighborhood. Clearly, $\mathbb{A}^n_S$ and $\mathbb{P}^n_S$ are smooth of relative dimension n for every scheme S. (It is harder to give examples of non-smooth schemes directly from the definition; we will come back to this later.)

(4.3) Dimension of schemes.

Recall from commutative algebra that for a ring R we define the (Krull) dimension $\dim R$ of R as the supremum over all lengths of chains of prime ideals, or equivalently as the dimension of the topological space $\operatorname{Spec} R$ in the sense of the following definition.

Definition 4.7. *Let X be a topological space. We define the dimension of X as*

$$\begin{aligned} \dim X := \sup\{\ell;\ &\textit{there exists a chain } Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_\ell \\ &\textit{of closed irreducible subsets } Z_i \subseteq X\}. \end{aligned}$$

We will use this notion of dimension for non-affine schemes, as well. Recall the following theorem about the dimension of finitely generated algebras over a field from commutative algebra:

Theorem 4.8. *Let k be a field, and let A be a finitely generated k-algebra which isi a domain. Let $\mathfrak{m} \subset A$ be a maximal ideal. Then*

$$\dim A = \operatorname{trdeg}_k(\operatorname{Frac}(A)) = \dim A_{\mathfrak{m}}.$$

By passing to an affine cover, we obtain the following corollary:

Corollary 4.9. *Let k be a field, and let X be an integral k-scheme which is of finite type over k. Denote by $K(X)$ its field of rational functions. Let $U \subseteq X$ be a non-empty open subset, and let $x \in X$ be a closed point. Then*

$$\dim X = \dim U = \operatorname{trdeg}_k(K(X)) = \dim \mathscr{O}_{X,x}.$$

(4.4) Existence of smooth points.

Let k be a field.

Lemma 4.10. *Let X, Y be [integral[2]] k-schemes which are locally of finite type over k. Let $x \in X$, $y \in Y$, and let $\varphi\colon \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ be an isomorphism of k-algebras. Then there exist open neighborhoods U of x and V of y and an isomorphism $h\colon U \to V$ of k-schemes with $h_x^\sharp = \varphi$.*

Proposition 4.11. *Let X be an integral k-scheme of finite type. Assume that $K(X) \cong k(T_1, \dots, T_d)[\alpha]$ with α separable algebraic over $k(T_1, \dots, T_d)$. (This is always possible if k is perfect.) (Then $\dim X = d$ by the above.)*

Then there exists a dense open subset $U \subseteq X$ and a separable irreducible polynomial $g \in k(T_1, \dots, T_d)[T]$ with coefficients in $k[T_1, \dots, T_d]$, such that U is isomorphic to a dense open subset of $\operatorname{Spec} k[T_1, \dots T_d]/(g)$.

Theorem 4.12. *Let k be a perfect field, and let X be a nonempty reduced k-scheme locally of finite type over k. Then the smooth locus*

$$X_{\rm sm} := \{x \in X;\ X \to \operatorname{Spec} k \textit{ is smooth in} x\}$$

of X is open and dense.

(4.5) Regular rings.

For references to the literature, see [GW1] App. B, in particular B.73, B.74, B.75

Definition 4.13. *A noetherian local ring A with maximal ideal $\mathfrak{m}$ and residue class field κ is called* regular*, if $\dim A = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.*

One can show that the inequality $\dim A \le \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ always holds. Therefore we can rephrase the definition as saying that A is regular if $\mathfrak{m}$ has a generating system consisting of $\dim A$ elements.

Definition 4.14. *A noetherian ring A is called* regular*, if $A_\mathfrak{m}$ is regular for every maximal ideal $\mathfrak{m} \subset A$.*

We quote the following (mostly non-trivial) results about regular rings:

Theorem 4.15.
(1) *Every localization of a regular ring is regular.*
(2) *If A is regular, then the polynomial ring $A[T]$ is regular.*
(3) *(Theorem of Auslander–Buchsbaum) Every regular local ring is factorial.*
(4) *Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and of dimension d, and let $f_1, \dots, f_r \in \mathfrak{m}$. Then $A/(f_1, \dots, f_r)$ is regular of dimension $d - r$ if and only if the images of the f_i in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $A/\mathfrak{m}$.*

[2]The statement is true in general, but in the lecture we proved it only with the additional assumption that X and Y are integral.

(4.6) Smoothness and regularity.

Let k be a field.

Lemma 4.16. *Let X be a k-scheme locally of finite type. Let $x \in X$ such that $X \to \operatorname{Spec} k$ is smooth of relative dimension d in x. Then $\mathscr{O}_{X,x}$ is regular of dimension $\le d$. If moreover x is closed, then $\mathscr{O}_{X,x}$ is regular of dimension d.*

Lemma 4.17. *Let $X = V(g_1, \dots, g_s) \subseteq \mathbb{A}^n_k$, and let $x \in X$ be a closed point. If $\operatorname{rk} J_{g_1,\dots,g_s}(x) \ge n - \dim \mathscr{O}_{X,x}$, then x is smooth in X/k, and $\operatorname{rk} J_{g_1,\dots,g_s}(x) = n - \dim \mathscr{O}_{X,x}$.*

Theorem 4.18. *Let X be a k-scheme locally of finite type, $x \in X$ a closed point, $d \ge 0$. Fix an algebraically closed extension field K of k and write $X_K = X \otimes_k K$. The following are equivalent:*
(i) *The morphism $X \to \operatorname{Spec} k$ is smooth of relative dimension d at x.*
(ii) *For all points $\overline{x} \in X_K$ lying over x, X_K is smooth over K of relative dimension d at $\overline{x}$.*
(iii) *There exists a point $\overline{x} \in X_K$ lying over x, such that X_K is smooth over K of relative dimension d at $\overline{x}$.*
(iv) *For all points $\overline{x} \in X_K$ lying over x, the local ring $\mathscr{O}_{X_K,\overline{x}}$ is regular of dimension d.*
(v) *There exists a point $\overline{x} \in X_K$ lying over x, such that the local ring $\mathscr{O}_{X_K,\overline{x}}$ is regular of dimension d.*

If these conditions hold, then the local ring $\mathscr{O}_{X,x}$ is regular of dimension d, and if $\kappa(x) = k$, then this last condition is equivalent to the previous ones.

Corollary 4.19. *Let X be an irreducible scheme of finite type over k, and let $x \in X(k)$ be a k-valued point. Then $X \to \operatorname{Spec} k$ is smooth at x if and only if $\dim X = \dim_k T_x X$.*

Corollary 4.20. *Let $X = V(g_1, \dots, g_s) \subseteq \mathbb{A}^n_k$ and let $x \in X$ be a smooth closed point. Let $d = \dim \mathscr{O}_{X,x}$. Then $J_{g_1,\dots,g_s}(x)$ has rank $n - d$. In particular, $s \ge n - d$.*

After renumbering the g_i, if necessary, there exists an open neighborhood U of x and an open immersion $U \subseteq V(g_1, \dots, g_{n-d})$, i.e., locally around x, "X is cut out in affine space by the expected number of equations".

Corollary 4.21. *Let X be locally of finite type over k. The following are equivalent:*
(i) *X is smooth over k.*
(ii) *$X \otimes_k L$ is regular for every field extension L/k.*
(iii) *There exists an algebraically closed extension field K of k such that $X \otimes_k K$ is regular.*

The sheaf of differentials.

General references: [M2] §25, [Bo] Ch. 8, [H] II.8.

(4.7) Modules of differentials.

Let A be a ring.

Definition 4.22. *Let B be an A-algebra, and M a B-module. An A-*derivation *from B to M is a homomorphism $D\colon B \to M$ of abelian groups such that*
(a) *(Leibniz rule) $D(bb') = bD(b') + b'D(b)$ for all $b, b' \in B$,*
(b) *$d(a) = 0$ for all $a \in A$.*

Assuming property (a), property (b) is equivalent to saying that D is a homomorphism of A-modules. We denote the set of A-derivations $B \to M$ by $\mathrm{Der}_A(B, M)$; it is naturally a B-module.

Definition 4.23. *Let B be an A-algebra. We call a B-module $\Omega_{B/A}$ together with an A-derivation $d_{B/A}\colon B \to \Omega_{B/A}$ a module of (relative, KÃ¤hler) differentials of B over A if it satisfies the following universal property:*

For every B-module M and every A-derivation $D\colon B \to M$, there exists a unique B-module homomorphism $\psi\colon \Omega_{B/A} \to M$ such that $D = \psi \circ d_{B/A}$.

In other words, the map $\mathrm{Hom}_B(\Omega_{B/A}, M) \to \mathrm{Der}_A(B, M)$, $\psi \mapsto \psi \circ d_{B/A}$ is a bijection.

Lemma 4.24. *Let I be a set, $B = A[T_i, i \in I]$ the polynomial ring. Then $\Omega_{B/A} := B^{(I)}$ with $d_{B/A}(T_i) = e_i$, the "i-th standard basis vector" is a module of differentials of B/A.*

So we can write $\Omega_{B/A} = \bigoplus_{i\in I} Bd_{B/A}(T_i)$.

Lemma 4.25. *Let $\varphi\colon B \to B'$ be a surjective homomorphism of A-algebras, and write $\mathfrak{b} = \mathrm{Ker}(\varphi)$. Assume that a module of differentials $(\Omega_{B/A}, d_{B/A})$ for B/A exists. Then*

$$\Omega_{B/A}/(\mathfrak{b}\Omega_{B/A} + B'd(\mathfrak{b}))$$

together with the derivation $d_{B'/A}$ induced by $d_{B/A}$ is a module of differentials for B'/A.

Corollary 4.26. *For every A-algebra B, a module $\Omega_{B/A}$ of differentials exists. It is unique up to unique isomorphism.*

We will see later that for a scheme morphism $X \to Y$, one can construct an $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with a "derivation" $\mathscr{O}_X \to \Omega_{X/Y}$ by gluing sheaves associated to modules of differentials attached to the coordinate rings of suitable affine open subschemes of X and Y.

Let $\varphi\colon A \to B$ be a ring homomorphism. For the next definition, we will consider the following situation: Let C be a ring, $I \subseteq C$ an ideal with $I^2 = 0$, and let

$$\begin{array}{ccc} B & \longrightarrow & C/I \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

be a commutative diagram (where the right vertical arrow is the canonical projection). We will consider the question whether for these data, there exists a homomorphism $B \to C$ (dashed in the following diagram) making the whole diagram commutative:

$$\begin{array}{ccc} B & \longrightarrow & C/I \\ {\scriptstyle\varphi}\uparrow & \searrow & \uparrow \\ A & \longrightarrow & C \end{array}$$

Definition 4.27. *Let $\varphi\colon A \to B$ be a ring homomorphism.*

(1) *We say that φ is* formally unramified*, if in every situation as above, there exists at most one homomorphism $B \to C$ making the diagram commutative.*
(2) *We say that φ is* formally smooth*, if in every situation as above, there exists at least one homomorphism $B \to C$ making the diagram commutative.*
(3) *We say that φ is* formally étale*, if in every situation as above, there exists a unique homomorphism $B \to C$ making the diagram commutative.*

Passing to the spectra of these rings, we can interpret the situation in geometric terms: $\operatorname{Spec} C/I$ is a closed subscheme of $\operatorname{Spec} C$ with the same topological space, so we can view the latter as an "infinitesimal thickening" of the former. The question becomes the question whether we can extend the morphism from $\operatorname{Spec} C/I$ to $\operatorname{Spec} B$ to a morphism from this thickening.

Proposition 4.28. *Let $\varphi\colon A \to B$ be a ring homomorphism. Then φ is formally unramified if and only if $\Omega_{B/A} = 0$.*

For an algebraic field extension L/K one can show that $K \to L$ is formally unramified if and only if it is formally smooth if and only if L/K is separable. Cf. Problem 27 and [M2] §25, §26 (where the discussion is extended to the general, not necessarily algebraic, case).

Theorem 4.29. *Let $f\colon A \to B$, $g\colon B \to C$ be ring homomorphisms. Then we obtain a natural sequence of C-modules*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

which is exact.

If moreover g is formally smooth, then the sequence

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

is a split short exact sequence.

Theorem 4.30. *Let $f\colon A \to B$, $g\colon B \to C$ be ring homomorphisms. Assume that g is surjective with kernel $\mathfrak{b}$. Then we obtain a natural sequence of C-modules*

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0,$$

where the homomorphism $\mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B C$ is given by $x \mapsto d_{B/A}(x) \otimes 1$.

If moreover $g \circ f$ is formally smooth, then the sequence

$$0 \to \mathfrak{b}/\mathfrak{b}^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

is a split short exact sequence.

(4.8) The sheaf of differentials of a scheme morphism.

Remark 4.31. Let again B an A-algebra. There is the following alternative construction of $\Omega_{B/A}$: Let $m\colon B \otimes_A B \to B$ be the multiplication map, and let $I = \mathrm{Ker}(m)$. Then I/I^2 is a B-module, and $d\colon B \to I/I^2$, $b \mapsto 1 \otimes b - b \otimes 1$, is an A-derivation. One shows that $(I/I^2, d)$ satisfies the universal property defining $(\Omega_{B/A}, d_{B/A})$.

We can use a similar definition as we used for ring homomorphisms above to define the notions of formally unramified, formally smooth and formally étale morphisms of schemes.

Definition 4.32. *Let $f\colon X \to Y$ be a morphism of schemes.*

(1) *We say that f is* formally unramified*, if for every ring C, every ideal I with $I^2 = 0$, and every morphism $\mathrm{Spec}\, C \to Y$ (which we use to view $\mathrm{Spec}\, C$ and $\mathrm{Spec}\, C/I$ as Y-schemes), the composition with the natural closed embedding $\mathrm{Spec}\, C/I \to \mathrm{Spec}\, C$ yields an injective map $\mathrm{Hom}_Y(\mathrm{Spec}\, C, X) \to \mathrm{Hom}_Y(\mathrm{Spec}\, C/I, X)$.*
(2) *We say that f is* formally smooth*, if for every ring C, every ideal I with $I^2 = 0$, and every morphism $\mathrm{Spec}\, C \to Y$, the composition with the natural closed embedding $\mathrm{Spec}\, C/I \to \mathrm{Spec}\, C$ yields a surjective map $\mathrm{Hom}_Y(\mathrm{Spec}\, C, X) \to \mathrm{Hom}_Y(\mathrm{Spec}\, C/I, X)$.*
(3) *We say that f is* formally étale*, if f is formally unramified and formally smooth.*

If f is a morphism of affine schemes, then f has one of the properties of this definition if and only if the corresponding ring homomorphism has the same property in the sense of our previous definition.

Lemma 4.33.

(1) *Every monomorphism of schemes (in particular: every immersion) is formally unramified.*
(2) *Let $A \to B \to C$ be ring homomorphisms such that $A \to B$ is formally unramified. Then we can naturally identify $\Omega_{C/A} = \Omega_{C/B}$.*

Definition 4.34. *Let $X \to Y$ be a morphism of schemes, and let $\mathscr{M}$ be an $\mathscr{O}_X$-module. A derivation $D\colon \mathscr{O}_X \to \mathscr{M}$ is a homomorphism of abelian sheaves such that for all open subsets $U \subseteq X$, $V \subseteq Y$ with $f(U) \subseteq V$, the map $\mathscr{O}(U) \to \mathscr{M}(U)$ is an $\mathscr{O}_Y(V)$-derivation.*

Equivalently, $D\colon \mathscr{O}_X \to \mathscr{M}$ is a homomorphism of $f^{-1}(\mathscr{O}_Y)$-modules such that for every open $U \subseteq X$, the Leibniz rule

$$D(U)(bb') = bD(U)(b') + b'D(U)(b), \qquad \forall b, b' \in \Gamma(U, \mathscr{O}_X)$$

holds.

We denote the set of all these derivations by $\mathrm{Der}_Y(\mathscr{O}_X, \mathscr{M})$; it is a $\Gamma(X, \mathscr{O}_X)$-module.

Definition/Proposition 4.35. *Let $f\colon X \to Y$ be a morphism of schemes. The following three definitions give the same result (up to unique isomorphism), called the sheaf of differentials of f or of X over Y, denoted $\Omega_{X/Y}$ — a quasi-coherent $\mathscr{O}_X$-module together with a derivation $d_{X/Y}\colon \mathscr{O}_X \to \Omega_{X/Y}$.*

(i) *There exists a unique $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with a derivation $d_{X/Y}\colon \mathscr{O}_X \to \Omega_{X/Y}$ such that for all affine open subsets $\mathrm{Spec}\, B = U \subseteq X$, $\mathrm{Spec}\, A = V \subseteq Y$ with $f(U) \subseteq V$, $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$ and $d_{X/Y|U}$ is induced by $d_{B/A}$.*

(ii) *$\Omega_{X/Y} = \Delta^*(\mathscr{J}/\mathscr{J}^2)$, where $\Delta\colon X \to X \times_Y X$ is the diagonal morphism, $W \subseteq X \times_Y X$ is open such that $\mathrm{im}(\Delta) \subseteq W$ is closed (if f is separated we can take $W = X \times_Y X$), and $\mathscr{J}$ is the quasi-coherent ideal defining the closed subscheme $\Delta(X) \subseteq W$. The derivation $d_{X/Y}$ is induced, on affine opens, by the map $b \mapsto 1 \otimes b - b \otimes 1$.*

(iii) *The quasi-coherent $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with $d_{X/Y}$ is characterized by the universal property that composition with $d_{X/Y}$ induces bijections*

$$\mathrm{Hom}_{\mathscr{O}_X}(\Omega_{X/Y}, \mathscr{M}) \xrightarrow{\sim} \mathrm{Der}_Y(\mathscr{O}_X, \mathscr{M})$$

for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$, functorially in $\mathscr{M}$.

The properties we proved for modules of differentials can be translated into statements for sheaves of differentials:

Proposition 4.36. *Let $f\colon X \to Y$, $g\colon Y' \to Y$ be morphisms of schemes, and let $X' = X \times_Y Y'$. Denote by $g'\colon X' \to X$ the base change of g. There is a natural isomorphism $\Omega_{X'/Y'} \cong (g')^*\Omega_{X/Y}$, compatible with the universal derivations.*

Proposition 4.37. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms of schemes. Then there is an exact sequence*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

of $\mathscr{O}_X$-modules. If f is formally smooth, then the sequence

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

is exact and splits locally on X.

Proposition 4.38. *Let $i\colon Z \to X$ be a closed immersion with corresponding ideal sheaf $\mathscr{J} \subseteq \mathscr{O}_X$, and let $g\colon X \to Y$ be a scheme morphism. Then there is an exact sequence*

$$i^*(\mathscr{J}/\mathscr{J}^2) \to i^*\Omega_{X/Y} \to \Omega_{Z/Y} \to 0$$

of $\mathscr{O}_Z$-modules. If Z is formally smooth over Y, then the sequence

$$0 \to i^*(\mathscr{J}/\mathscr{J}^2) \to i^*\Omega_{X/Y} \to \Omega_{Z/Y} \to 0$$

is exact and splits locally on Z.

Proposition 4.39. *Let K be a field, and let X be a k-scheme of finite type. Let $x \in X(k)$. Then we have an isomorphism $T_xX = \Omega_{X/k}(x)$ between the Zariski tangent space at x and the fiber of the sheaf of differentials of X/k at x.*

Proposition 4.40. *Let R be a ring. We have a short exact sequence*

$$0 \to \Omega_{\mathbb{P}^n_R/R} \to \mathscr{O}(-1)^{n+1} \to \mathscr{O}^n \to 0$$

of $\mathscr{O}_X$-modules.

(4.9) Sheaves of differentials and smoothness.

We start by slightly rephrasing the definition of a smooth morphism.

Definition 4.41. *A morphism $f\colon X \to Y$ of schemes is called* smooth of relative dimension $d \geq 0$ in $x \in X$, *if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \operatorname{Spec} R \subseteq Y$ of $f(x)$ such that $f(U) \subseteq V$ and an open immersion $j\colon U \to \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d})$ such that the triangle*

$$\begin{array}{ccc} U & \xrightarrow{\;j\;} & \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d}) \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & V & \end{array}$$

is commutative, and that the images of df_1, ..., df_{n-d} in the fiber $\Omega^1_{\mathbb{A}^n_R/R} \otimes \kappa(x)$ are linearly independent over $\kappa(x)$. (We view x as a point of $\mathbb{A}^n_R$ via the embedding $U \to \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d}) \to \operatorname{Spec} R[T_1, \dots, T_n] = \mathbb{A}^n_R$.)

Proposition 4.42. *Let $f\colon X \to S$ be smooth of relative dimension at $x \in X$. Then there exists an open neighborhood U of x such that the restriction $\Omega_{X/Y|U} (= \Omega_{U/Y})$ is free of rank d.*

Theorem 4.43. *Let k be an algebraically closed field, and let X be an irreducible k-scheme of finite type. Let $d = \dim X$. Then X is smooth over k if and only if $\Omega_{X/k}$ is locally free of rank d.*

Proposition 4.44. *Let $f\colon X \to S$ be smooth of relative dimension d at $x \in X$. Then there exists an open neighborhood U of x such that the restriction $U \to S$ of f to U is formally smooth.*

Theorem 4.45. *Let $f\colon X \to Y$ be a morphism locally of finite presentation (e.g., if Y is noetherian and f is locally of finite type). Then f is smooth if and only if f is formally smooth.*

We skip the proof that smoothness implies formal smoothness, see for instance [Bo] Ch. 8.5. (But cf. the previous proposition which shows that a smooth morphism is at least "locally formally smooth".)

5. Projective schemes *

(5.1) The Proj construction.

Reference: [GW1] Ch. 13.

Definition 5.1.

(1) *A* graded ring *is a ring A with a decomposition $A = \bigoplus_{d\geq 0} A_d$ as abelian groups such that $A_d \cdot A_e \subseteq A_{d+e}$ for all d, e. The elements of A_d are called* homogeneous of degree d.

(2) *Let R be a ring. A* graded R-algebra *is a graded ring A together with a ring homomorphism $R \to A$.*

(3) *A* homomorphism $A \to B$ of graded rings (or graded R-algebras) *is a ring homomorphism (or R-algebra homomorphism, respectively) $f\colon A \to B$ such that $f(A_d) \subseteq B_d$ for all d.*

(4) *Let A be a graded ring. A* graded A-module *is an A-module M with a decomposition $M = \bigoplus_{d\in\mathbb{Z}} M_d$ such that $A_d \cdot M_e \subseteq M_{d+e}$ for all d, e. The elements of M_d are called* homogeneous of degree d.

(5) *A* homomorphism $M \to N$ of graded A-modules *is an A-module homomorphism $f\colon M \to N$ such that $f(M_d) \subseteq N_d$ for all d.*

(6) *Let A be a graded ring and let M be a graded A-module. A* homogeneous submodule of M *is a submodule $N \subseteq M$ such that $N = \bigoplus_{d\in\mathbb{Z}}(N \cap M_d)$. In this way, N is itself a graded A-module and the inclusion $N \hookrightarrow M$ is a homomorphism of graded A-modules. (And conversely, every injective homomorphism of graded A-modules has a homogeneous submodule as its image.) A homogeneous submodule of A is called a* homogeneous ideal.

Example 5.2. Let R be a ring. Then the polynomial ring $R[T_0, \dots, T_n]$ is a graded R-algebra if we set $R[T_0, \dots, T_n]_d$ to be the R-submodule of homogeneous polynomials of degree d.

We now fix a graded ring A.

For a homogeneous element $f \in A_e$, and a graded A-module M, the localization M_f is a graded A-module via

$$M_{f,d} = \{\frac{m}{f^i};\ m \in M_{d+ei}\}.$$

Applying this to A as an A-module, we obtain a grading on A_f giving A_f the structure of a graded ring. Then M_f is a graded A_f-module.

We define

$$M_{(f)} := M_{f,0},$$

the degree 0 part of M_f. Then $A_{(f)}$ is a ring and $M_{(f)}$ is an $A_{(f)}$-module.

Example 5.3. Let R be a ring. Then

$$R[T_0, \dots, T_n]_{(T_i)} = R[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}].$$

Definition 5.4. *We write $A_+ := \bigoplus_{d>0} A_d$, an ideal of A. A homogeneous prime ideal $\mathfrak{p} \subset A$ is called* relevant *if $A_+ \not\subseteq \mathfrak{p}$.*

Definition 5.5. *We denote by $\operatorname{Proj}(A)$ the set of all relevant homogeneous prime ideals of A. We equip $\operatorname{Proj}(A)$ with the Zariski topology, by saying that the closed subsets are the subsets of the form*

$$V_+(I) := \{\mathfrak{p} \in \operatorname{Proj}(A);\ I \subseteq \mathfrak{p}\}.$$

for homogeneous ideals $I \subseteq A$.

For a homogeneous element f, we write $D_+(f) := \operatorname{Proj}(A) \setminus V_+(f)$.

Lemma 5.6. *Let $f \in A$ be a homogeneous element. Then the map*

$$D_+(f) \to \operatorname{Spec} A_{(f)}, \quad \mathfrak{p} \mapsto (\mathfrak{p}A_f) \cap A_{(f)}$$

is a homeomorphism.

Proposition 5.7. *There is a unique sheaf $\mathscr{O}$ of rings on $\operatorname{Proj}(A)$ such that*

$$\Gamma(D_+(f), \mathscr{O}) = A_{(f)}$$

for every homogeneous element $f \in A$ and with restriction maps given by the canonical maps between the localizations. The ringed space $(\operatorname{Proj}(A), \mathscr{O})$ is a separated scheme which we again denote by $\operatorname{Proj}(A)$.

Definition 5.8. *Let R be a ring, and let X be an R-scheme. We say that X is* projective over R *(or that the morphism $X \to \operatorname{Spec} R$ is* projective*), if there exist $n \geq 0$ and a closed immersion $X \to \mathbb{P}^n_R$ of R-schemes.*

Theorem 5.9. *Let R be a ring, and let X be a projective R-scheme. Then X is proper over R.*

(5.2) Quasi-coherent modules on $\operatorname{Proj}(A)$.

Let A be a graded ring, $X = \operatorname{Proj} A$. If M is a graded A-module, there is a unique sheaf $\widetilde{M}$ of $\mathscr{O}_X$-modules such that

$$\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$$

for every homogeneous element $f \in A$, and such that the restriction maps for inclusions of the form $D_+(g) \subseteq D_+(f)$ are given by the natural maps between the localizations. This sheaf is a quasi-coherent $\mathscr{O}_X$-module.

Example 5.10. Let $A(n)$ be the graded A-module defined by $A(n) = \bigoplus_{d\in\mathbb{Z}} A_{n+d}$. We set $\mathscr{O}_X(n) = \widetilde{A(n)}$. If $A = R[T_0, \dots, T_n]$ for a ring R, so that $X = \mathbb{P}^n_R$, then this notation is consistent with our previous definition.

For $f \in A_d$, multiplication by f^k defines an isomorphism

$$\mathscr{O}_{X|D_+(f)} \xrightarrow{\sim} \mathscr{O}_X(n)_{|D_+(f)}.$$

In particular, if A is generated as an A_0-algebra by A_1, then $\mathscr{O}_X(n)$ is a line bundle.

Assume, for the remainder of this section, that A is generated as an A_0-algebra by A_1. So $X = \bigcup_{f\in A_1} D_+(f)$, and $\mathscr{O}_X(n)$ is a line bundle.

For an $\mathscr{O}_X$-module $\mathscr{F}$, write $\mathscr{F}(n) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$, and define a graded A-module $\Gamma_*(\mathscr{F})$ by

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathscr{F}(n)).$$

Lemma 5.11. *For a graded A-module M, there is a natural map $M \to \Gamma_*(\widetilde{M})$. For an $\mathscr{O}_X$-module $\mathscr{F}$, there is a natural map $\widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$. If $\mathscr{F}$ is quasi-coherent, then the latter map is an isomorphism.*

Call a graded A-module M *saturated*, if the map $M \to \Gamma_*(\widetilde{M})$ is an isomorphism.

Proposition 5.12. *The functors $M \to \widetilde{M}$ and $\mathscr{F} \to \Gamma_*(\mathscr{F})$ define an equivalence of categories between the category of saturated graded A-modules and the category of quasi-coherent $\mathscr{O}_X$-modules.*

6. Cohomology of $\mathscr{O}_X$-modules

General references: [We], [HS], [Gr], [KS], [GW2].

The formalism of derived functors.

(6.1) Complexes in abelian categories.

Reference: [We] Ch. 1, [GW2] App. F.

Let $\mathcal{A}$ be an abelian category (e.g., the category of abelian groups, the category of R-modules for a ring R, the category of abelian sheaves on a topological space X, the category of $\mathscr{O}_X$-modules on a ringed space X, or the category of quasi-coherent $\mathscr{O}_X$-modules on a scheme X; see [GW2] Section (F.7)).

A *complex in* $\mathcal{A}$ is a sequence of morphisms

$$\cdots \longrightarrow A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \longrightarrow \cdots$$

in $\mathcal{A}$ ($i \in \mathbb{Z}$), such that $d^{i+1} \circ d^i = 0$ for every $i \in \mathbb{Z}$. The maps d^i are called the *differentials* of the complex.

Given complexes $A^\bullet$, $B^\bullet$, a morphism $A^\bullet \to B^\bullet$ of complexes is a family of morphisms $f^i\colon A^i \to B^i$ such that the f^i commute with the differentials of $A^\bullet$ and $B^\bullet$ in the obvious way. With this notion of morphisms, we obtain the category $C(\mathcal{A})$ of complexes in $\mathcal{A}$. This is an abelian category (kernels, images, …are formed degree-wise); see [We] Thm. 1.2.3.

Definition 6.1. *Let $A^\bullet$ be a complex in $\mathcal{A}$. For $i \in \mathbb{Z}$, we call*

$$H^i(A^\bullet) := \operatorname{Ker}(d^i)/\operatorname{im}(d^{i-1})$$

the i-th cohomology object *of $A^\bullet$. We obtain functors $H^i\colon C(\mathcal{A}) \to \mathcal{A}$. We say that $A^\bullet$ is* exact at i, *if $H^i(A^\bullet) = 0$. We say that $A^\bullet$ is* exact, *if $H^i(A^\bullet) = 0$ for all i.*

Remark 6.2. Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a sequence of morphisms of complexes. The sequence is exact (in the sense that at each point the kernel and image in the category $C(\mathcal{A})$ coincide) if and only if for every i, the sequence $0 \to A^i \to B^i \to C^i \to 0$ is exact.

Proposition 6.3. [We] Thm. 1.3.1 *Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be an exact sequence of complexes in $\mathcal{A}$. Then there are maps $\delta^i\colon H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ (called* boundary maps*) such that, together with the maps induced by functoriality of the H^i, we obtain the* long exact cohomology sequence

$$\cdots H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots .$$

We need a criterion which ensures that two morphisms between complexes induce the same maps on all cohomology objects. See [We] 1.4.

Definition 6.4. *Let $f, g\colon A^\bullet \to B^\bullet$ be morphisms of complexes. We say that f and g are* homotopic*, if there exists a family of maps $k^i\colon A^i \to B^{i-1}$, $i \in \mathbb{Z}$, such that*

$$f - g = dk + kd,$$

which we use as short-hand notation for saying that for every i,

$$f^i - g^i = d_B^{i-1} \circ k^i + k^{i+1} \circ d_A^i.$$

In this case we write $f \sim g$. The family $(k^i)_i$ is called a homotopy*.*

Proposition 6.5. *Let $f, g\colon A^\bullet \to B^\bullet$ be morphisms of complexes which are homotopic. Then for every i, the maps $H^i(A^\bullet) \to H^i(B^\bullet)$ induced by f and g are equal.*

In particular, if $A^\bullet$ is a complex such that $\mathrm{id}_{A^\bullet} \sim 0$, then $H^i(A^\bullet) = 0$ for all i, i.e., $A^\bullet$ is exact.

Definition 6.6. *Let $A^\bullet$ and $B^\bullet$ be complexes. We say that $A^\bullet$ and $B^\bullet$ are* homotopy equivalent*, if there exist morphisms $f\colon A^\bullet \to B^\bullet$ and $g\colon B^\bullet \to A^\bullet$ of complexes such that $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_B$. In this case, f and g induce isomorphisms $H^i(A^\bullet) \cong H^i(B^\bullet)$ for all i.*

(6.2) Left exact functors.

Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. All functors $F\colon \mathcal{A} \to \mathcal{B}$ that we consider are assumed to be additive, i.e., they induce group homomorphisms $\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(A'))$ for all A, A' in $\mathcal{A}$.

Definition 6.7. *A (covariant) functor $F\colon \mathcal{A} \to \mathcal{B}$ is called* left exact*, if for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, the sequence*

$$0 \to F(A') \to F(A) \to F(A'')$$

is exact.

Definition 6.8. *A contravariant functor $F\colon \mathcal{A} \to \mathcal{B}$ is called* left exact *if for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, the sequence*

$$0 \to F(A'') \to F(A) \to F(A')$$

is exact.

Similarly, we have the notion of right exact functor. A functor which is left exact and right exact (and hence preserves exactness of arbitrary sequences) is called exact.

Let $A_0 \in \mathcal{A}$. Then the functors $A \mapsto \mathrm{Hom}_{\mathcal{A}}(A, A_0)$ and $A \mapsto \mathrm{Hom}_{\mathcal{A}}(A_0, A)$ are left exact.

(6.3) δ-functors.

Reference: [We] 2.1, [GW2] Section (F.48).

Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories.

Definition 6.9. *A* δ-functor *from $\mathcal{A}$ to $\mathcal{B}$ is a family $(T^i)_{i\geq 0}$ of functors $\mathcal{A} \to \mathcal{B}$ together with morphisms $\delta^i\colon T^i(A'') \to T^{i+1}(A')$ (called* boundary morphisms*) for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, such that the sequence*

$$0 \to T^0(A') \to T^0(A) \to T^0(A'') \to T(A') \to \cdots$$

is exact, and such that the δ^i are compatible with morphisms of short exact sequences in the obvious way.

Definition 6.10. *A δ-functor $(T^i)_i$ from $\mathcal{A}$ to $\mathcal{B}$ is called* universal*, if for every δ-functor $(S^i)_i$ and every morphism $f^0\colon T^0 \to S^0$ of functors, there exist unique morphisms $f^i\colon T^i \to S^i$ of functors for al $i > 0$, such that the f^i, $i \geq 0$ are compatible with the boundary maps δ^i of the two δ-functors for each short exact sequence in $\mathcal{A}$.*

The definition implies that given a (left exact) functor F, any two universal δ-functors $(T^i)_i$, $(T'^i)_i$ with $T^0 = T'^0 = F$ are isomorphic (in the obvious sense) via a unique isomorphism.

Definition 6.11. *A functor $F\colon \mathcal{A} \to \mathcal{B}$ is called* effaceable*, if for every X in $\mathcal{A}$ there exists a monomorphism $\iota\colon X \hookrightarrow A$ with $F(\iota) = 0$.*

A particular case is the situation where each X admits a monomorphism to an object I with $F(I) = 0$.

Proposition 6.12. ([We] Thm. 2.4.7, Ex. 2.4.5.) *Let $(T^i)_i$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$ such that for every $i > 0$, the functor T^i is effaceable. Then $(T^i)_i$ is a universal δ-functor.*

(6.4) Injective objects.

Let $\mathcal{A}$ be an abelian category.

Definition 6.13. *An object I in $\mathcal{A}$ is called* injective*, if the functor $X \mapsto \mathrm{Hom}_{\mathcal{A}}(X, I)$ is exact.*

If I is injective, then every short exact sequence $0 \to I \to A \to A'' \to 0$ in $\mathcal{A}$ splits. Conversely, if I is an object such that every short exact sequence $0 \to I \to A \to A'' \to 0$ splits, then I is injective.

Definition 6.14. *Let $X \in \mathcal{A}$. An* injective resolution *of X is an exact sequence*

$$0 \to X \to I^0 \to I^1 \to I^2 \to \cdots$$

in $\mathcal{A}$, where every I^i is injective.

Definition 6.15. *We say that the category $\mathcal{A}$* has enough injectives *if for every object X there exists a monomorphism $X \hookrightarrow I$ from X into an injective object I. Equivalently: Every object has an injective resolution.*

Remark 6.16.

(1) The categories of abelian groups, of R-modules for a ring R, of abelian sheaves on a topological space, and more generally of $\mathscr{O}_X$-modules on a ringed space X all have enough injective objects. For these categories, it is not too hard to show this statement directly (for $\mathscr{O}_X$-modules on a ringed space X, e.g., see [H] Proposition III.2.2). Alternatively, one can show that they all are "Grothendieck abelian categories", and that those have enough injective objects (see [GW2] Sections (F.12), (21.2)).

(2) Dually, we have the notion of *projective* object (i.e., P such that $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact), of *projective resolution* $\cdots \to P_1 \to P_0 \to A \to 0$, and of abelian categories with *enough projective objects*. For a ring R, the category of R-modules clearly has enough projectives, since every free module is projective, and every module admits an epimorphism from a free module. Categories of sheaves of abelian groups or $\mathscr{O}_X$-modules typically do not have enough projectives.

Lemma 6.17. ([We] Theorem 2.3.7, see also Porism 2.2.7) *Let $\mathcal{A}$ be an abelian category and let $f\colon A \to B$ be a morphism in $\mathcal{A}$. Let $I^i \in \mathcal{A}$ be injective, $i \geq 0$, and let $0 \to A \to A^0 \to A^1 \to \cdots$ be exact with $A^i \in \mathcal{A}$. Then there exists a morphism $g\colon A^\bullet \to I^\bullet$ of complexes such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \end{array}$$

is commutative, and g is uniquely determined up to homotopy.

Lemma 6.18. (cf. [We] Lemma 2.2.8, "Horseshoe lemma") *Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence in an abelian category $\mathcal{A}$. Let $0 \to A_1 \to I_1^\bullet$ and $0 \to A_3 \to I_3^\bullet$ be injective resolutions. Let $I_2^i := I_1^i \times I_3^i$ and define $A_2 \to I_2^0$ by using the composition $A_2 \to A_3 \to I_3^0$ and lifting the map $A_1 \to I_1^0$ to a map $A_3 \to I_0^1$, using the injectivity of I_0^1. Then we obtain an injective resolution $0 \to A_2 \to I_2^\bullet$, and a term-wise split short exact sequence*

$$0 \longrightarrow I_1^\bullet \longrightarrow I_2^\bullet \longrightarrow I_3^\bullet \longrightarrow 0$$

of complexes.

(6.5) Right derived functors.

Theorem 6.19. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor, and assume that $\mathcal{A}$ has enough injectives.*

For each $A \in \mathcal{A}$, fix an injective resolution $0 \to A \to I^\bullet$, and define

$$R^iF(A) = H^i(F(I^\bullet)), \qquad i \geq 0,$$

where $F(I^\bullet)$ denotes the complex obtained by applying the functor to all I^i and to the differentials of the complex $I^\bullet$. Then:

(1) *The R^iF are additive functors $\mathcal{A} \to \mathcal{B}$, and R^iFX is independent of the choice of injective resolution of X up to natural isomorphism.*
(2) *We have an isomorphism $F \cong R^0F$ of functors.*
(3) *For I injective, we have $R^iFI = 0$ for all $i > 0$.*
(4) *The family $(R^iF)_i$ admits natural boundary maps making it a universal δ-functor.*

We call the R^iF the right derived functors *of F.*

Remarks on the proof. For part (1), the main ingredient is Lemma 6.17. Parts (2) and (3) are easy. To construct the boundary maps giving rise to long exact cohomology sequences in Part (4), use Lemma 6.18; since the exact sequence $0 \to I_1^\bullet \to I_2^\bullet \to I_3^\bullet \to 0$ is term-wise split, the exactness is preserved when applying the functor F. Finally, this δ-functor is effaceable by (c) and hence universal. □

Definition 6.20. *Let F be a left exact functor as above. We say that an object $A \in \mathcal{A}$ is F-*acyclic*, if $R^iF(A) = 0$ for all $i > 0$.*

Definition 6.21. *Let F be a left exact functor as above, and let $A \in \mathcal{A}$. An F-*acyclic resolution *of A is an exact sequence $0 \to A \to J^0 \to J^1 \to \cdots$ where all J^i are F-acyclic.*

Proposition 6.22. *Let F be a left exact functor as above, and let $A \in \mathcal{A}$. Let $0 \to A \to J^0 \to J^1 \to \cdots$ be an F-acyclic resolution. Then we have natural isomorphisms $R^iF(A) = H^i(F(J^\bullet))$, i.e., we can compute $R^iF(A)$ by an F-acyclic resolution.*

Sketch of the proof. The assertion is easy to see for $i = 0$. For $i = 1$, note that $\mathrm{Ker}(F(J^1) \to F(J^2)) = F(J^0/A)$ by the left exactness of F. From this, one gets $R^1F(A) = H^1(F(J^\bullet))$.

From the long exact cohomology sequence attached to the short exact sequence $0 \to A \to J^0 \to J^0/A \to 0$, we get isomorphisms $R^iF(J^0/A) \cong R^{i+1}F(A)$ for $i \geq 1$. Since $0 \to J^0/A \to J^1 \to \cdots$ is an acyclic resolution of J^0/A, we get

$$R^{i+1}F(A) \cong R^iF(J^0/A) \cong H^{i+1}(F(J^\bullet)),$$

where the second isomorphism holds by induction. □

(6.6) Derived categories.

Sometimes it is useful to employ the language of derived categories which gives rise to a notion of derived functor RF that contains slightly more information than the family $(R^iF)_i$ discussed above (for a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$). In fact, RF attaches to each object of $\mathcal{A}$, and more generally to each complex A in $C(\mathcal{A})$, a *complex* $RF(A)$ (up to some equivalence relation; more precisely, an object $RF(A) \in D(\mathcal{B})$ in the derived category of $\mathcal{B}$) whose cohomology objects are the $R^iF(A)$, i.e., $H^i(RF(A)) = R^iF(A)$.

See [GW2] (and the references given there) for a systematic treatment: Appendix F for the general theory, in particular Sections (F.37) for the definition of the derived category of an abelian category, (F.42), (F.43), (F.44) for the notion of derived functor, (F.48) for a comparison with the notion of δ-functor defined above, Chapter 21 for general results on cohomology of $\mathscr{O}_X$-modules on a ringed space X (specifically, the derived functors of the global section and direct image functors), Chapter 22 for cohomology of quasi-coherent $\mathscr{O}_X$-modules on a scheme X, Chapter 23 for cohomology of proper schemes and the derived direct image functor of a proper morphism. While in these chapters one sometimes obtains stronger or clearer statements using derived categories, for a large part the key arguments are exactly the same as with the “naive approach” using derived functors that we will take. In Chapter 24 on the theorem on formal functions and in particular in Chapter 25 on Grothendieck duality, the machinery of derived categories really shows its full power (but it is unlikely that we will get there in this class).

Cohomology of sheaves.

General reference: [H] Ch. III, [Stacks], [GW2] Chapter 21.

(6.7) Cohomology groups.

Let X be a topological space. Denote by (Ab_X) the category of abelian sheaves (i.e., sheaves of abelian groups) on X. We have the global section functor

$$\Gamma\colon (\mathrm{Ab}_X) \to (\mathrm{Ab}), \quad \mathscr{F} \mapsto \Gamma(X, \mathscr{F}),$$

to the category of abelian groups. This is a left exact functor, and we denote its right derived functors by $H^i(X, -)$. We call $H^i(X, \mathscr{F})$ the *i-th cohomology group of X with coefficients in $\mathscr{F}$.*

Example 6.23. For a field k, $H^1(\mathbb{P}^1_k, \mathscr{O}(-2)) \neq 0$.

(6.8) Flasque sheaves.

Reference: [GW2] Section (21.7).

Definition 6.24. *Let X be a topological space. A sheaf $\mathscr{F}$ on X is called* flasque *(or* flabby*), if all restriction maps $\mathscr{F}(U) \to \mathscr{F}(V)$ for $V \subseteq U \subseteq X$ open are surjective.*

For the next lemma, we use the *extension by zero* functor: Let $j\colon U \to X$ be the inclusion of an open subspace into a topological space X (similarly one can work with an open immersion of ringed spaces). For an abelian sheaf $\mathscr{F}$ on U we define $j_!\mathscr{F}$ as the sheafification of the presheaf

$$V \mapsto \begin{cases} \mathscr{F}(V) & \text{if } V \subseteq U, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain a left adjoint to the restriction functor j^*.

Lemma 6.25. *Let X be a ringed space. Let $\mathscr{F}$ be an injective object in the category of $\mathscr{O}_X$-modules. Then $\mathscr{F}$ is flasque.*

Sketch of proof. Identify $\mathscr{F}(U) = \operatorname{Hom}(j_{U,!}\mathscr{O}_U, \mathscr{F})$, where $j_U\colon U \to X$ is the inclusion of an open subset U into X, and $j_{U,!}$ is the extension by zero functor, using the adjunction between $j_{U,!}$ and j_U^*. □

Proposition 6.26. *Let X be a topological space, and let $\mathscr{F}$ be a flasque abelian sheaf on X. Then $\mathscr{F}$ is Γ-acyclic, i.e., $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.*

Proof. Given a flasque sheaf $\mathscr{F}$; embed it into an injective abelian sheaf and use the results of Problem 40 and dimension shift. □

Corollary 6.27. *Let X be a ringed space. The right derived functors of the global section functor from the category of $\mathscr{O}_X$-modules to the category of abelian groups can naturally be identified with $H^i(X, -)$.*

It follows that for an $\mathscr{O}_X$-module $\mathscr{F}$ the cohomology groups $H^i(X, \mathscr{F})$ carry a natural $\Gamma(X, \mathscr{O}_X)$-module structure.

(6.9) Grothendieck vanishing.

Reference: [H] III.2, [GW2] Section (21.12).

Lemma 6.28. *Let X be a topological space, and let $\iota\colon Y \to X$ be the inclusion of a closed subset Y of X. Let $\mathscr{F}$ be an abelian sheaf on X. Then there are natural isomorphisms*

$$H^i(Y, \mathscr{F}) = H^i(X, \iota_*\mathscr{F}), \quad i \geq 0.$$

Sketch of proof. Use that ι_* preserves the property of being flasque, and that the cohomology groups can be computed using a flasque resolution. □

Proposition 6.29. *Let X be a noetherian topological space, and let $(\mathscr{F}_i)_i$ be a filtered inductive system of abelian sheaves on X. Then the natural homomorphism*

$$\operatorname{colim}_i H^n(X, \mathscr{F}_i) \to H^n(X, \operatorname{colim}_i \mathscr{F})$$

is an isomorphism for all $n \geq 0$.

Sketch of proof. First show the statement for global sections, i.e., for $n = 0$. Since filtered colimits are exact, both sides give δ-functors from the abelian category of inductive systems of abelian sheaves to the category of abelian groups. To conclude the proof, it is enough to show that they are both effaceable (because this implies they are universal, and universal δ-functors are determined by their 0-th term).

To show effaceability, take a system $(\mathscr{F}_i)_i$ and consider functorial flasque resolutions of the $\mathscr{F}_i$ (such as the Godement resolution). Since filtered colimits are exact and, on noetherian spaces, preserve flasqueness, this gives rise to a flasque resolution of $\operatorname{colim} \mathscr{F}_i$. With this at hand, it is not hard to finish the proof. □

Let X be a topological space, $U \subseteq X$ open and $Z = X \setminus U$ its closed complement. Denote by $j\colon U \to X$ and $i\colon Z \to X$ the inclusions. Recall the *extension by zero* functor $j_!\colon (\mathrm{Ab}_U) \to (\mathrm{Ab}_X)$ (which is left adjoint to the pull-back j^{-1}). From this adjunction and the one between i^* and i_* we obtain the maps in the short exact sequence

$$0 \to j_!(\mathscr{F}_{|U}) \to \mathscr{F} \to i_*(\mathscr{F}_{|Z}) \to 0$$

of abelian sheaves on X. The exactness can be checked on stalks where it is clear.

Theorem 6.30. (Grothendieck) *Let X be a noetherian topological space (i.e., the descending chain condition holds for closed subsets of X), let $n = \dim X$, and let $\mathscr{F}$ be a sheaf of abelian groups on X. Then*

$$H^i(X, \mathscr{F}) = 0 \quad \textit{for all } i > n.$$

Proof. The proof is omitted in these notes. It consists of a series of reduction steps using the above ingredients, ultimately reducing to the fact that the cohomology of the constant sheaf $\mathbb{Z}_X$ on an irreducible space vanishes in positive degrees (since there a constant sheaf is flasque). See the references given above for more. □

Čech cohomology.

Reference: [GW2] Sections (21.14) to (21.17), [H] III.4, [Stacks] `01ED` (and following sections); a classical reference is [Go].

(6.10) Čech cohomology groups.

Let X be a topological space, and let $\mathscr{F}$ be an abelian sheaf on X. (The definitions below can be made more generally for presheaves.)

Let $\mathscr{U} = (U_i)_{i\in I}$ be an open cover of X. We fix a total order of the index set I (but see below for a sketch that the results are independent of this). For $i_0, \dots, i_p \in I$, we write $U_{i_0\dots i_p} := \bigcap_{\nu=0}^p U_{i_\nu}$.

We define

$$C^p(\mathscr{U}, \mathscr{F}) = \prod_{i_0<\dots<i_p} \Gamma(U_{i_0\dots i_p}, \mathscr{F})$$

and

$$d\colon C^p(\mathscr{U}, \mathscr{F}) \to C^{p+1}(\mathscr{U}, \mathscr{F}), \quad (s_{\underline{i}})_{\underline{i}} \mapsto \left(\sum_{\nu=0}^{p+1} (-1)^\nu s_{i_0\dots\widehat{i_\nu}\dots i_p|U_{\underline{i}}}\right)_{\underline{i}},$$

where $\widehat{\cdot}$ indicates that the corresponding index is omitted. One checks that $d \circ d = 0$, so we obtain a complex, the so-called *Čech complex for the cover $\mathscr{U}$ with coefficients in $\mathscr{F}$.*

Definition 6.31. *The* Čech cohomology groups *for $\mathscr{U}$ with coefficients in $\mathscr{F}$ are defined as*

$$\check{H}^p(\mathscr{U}, \mathscr{F}) = H^p(C^\bullet(\mathscr{U}, \mathscr{F})), \quad p \geq 0.$$

Since $\mathscr{F}$ is a sheaf, we have $\check{H}^0(\mathscr{U}, \mathscr{F}) = \Gamma(X, \mathscr{F}) = H^0(X, \mathscr{F})$. In fact, a presheaf $\mathscr{F}$ is a sheaf if and only if for all open subsets $U \subseteq X$ and all covers $\mathscr{U}$ of U the natural map $\Gamma(U, \mathscr{F}) \to \check{H}^0(\mathscr{U}, \mathscr{F})$ is an isomorphism.

(6.11) The "full" Čech complex.

Instead of the "alternating" (or "ordered") Čech complex as above, we can also consider the "full" Čech complex

$$C^p_f(\mathscr{U}, \mathscr{F}) = \prod_{i_0,\dots,i_p} \Gamma(U_{i_0\dots i_p}, \mathscr{F}),$$

with differentials defined by the same formula as above. Then the projection $C^\bullet_f(\mathscr{U}, \mathscr{F}) \to C^\bullet(\mathscr{U}, \mathscr{F})$ is a homotopy equivalence, with "homotopy inverse" given by

$$(s_{\underline{i}})_{\underline{i}} \mapsto (t_{\underline{i}})_{\underline{i}},$$

where $t_{\underline{i}} = 0$ whenever two entries in the multi-index $\underline{i}$ coincide, and otherwise $t_{\underline{i}} = \operatorname{sgn}(\sigma) s_{\sigma(\underline{i})}$, where σ is the permutation such that $\sigma(\underline{i})$ is in increasing order.

In particular, we have natural isomorphisms between the cohomology groups of the two complexes. So we also see that the Čech cohomology groups as defined above are independent of the choice of order on I.

(6.12) Passing to refinements.

Definition 6.32. *A* refinement *of a cover $\mathscr{U} = (U_i)_i$ of X is a cover $\mathscr{V} = (V_j)_{j\in J}$ (with J totally ordered) together with a map $\lambda\colon J \to I$ respecting the orders on I and J such that $V_j \subseteq U_{\lambda(j)}$ for every $j \in J$.*

Given a refinement $\mathscr{V}$ of $\mathscr{U}$, one obtains a natural map (using restriction of sections to smaller open subsets)

$$\check{H}^p(\mathscr{U}, \mathscr{F}) \to \check{H}^p(\mathscr{V}, \mathscr{F}).$$

We can pass to the colimit over all these maps given by refinements, and define

$$\check{H}^p(X, \mathscr{F}) := \operatorname*{colim}_{\mathscr{U}} \check{H}^p(\mathscr{U}, \mathscr{F}),$$

the *p-th Čech cohomology group of X with coefficients in $\mathscr{F}$*. More precisely, we here take the colimit over the category of refinements (see [GW2] Section (F.3)). Working with the full Čech complex, one can view this as a colimit over a partially ordered set "as usual", for the cohomology groups and even for the Čech complexes, if one considers them in the homotopy category (i.e., the map between complexes attached to a refinement is independent of λ up to homotopy), cf. [GW2] Section (21.16), also for the discussion of set-theoretic issues.

Proposition 6.33. *Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be a short exact sequence of abelian sheaves on X. Then there exists a homomorphism $\delta\colon \Gamma(X, \mathscr{F}'') \to \check{H}^1(X, \mathscr{F}')$ such that the sequence*

$$\begin{aligned} 0 \to &\Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \\ \to &\check{H}^1(X, \mathscr{F}') \to \check{H}^1(X, \mathscr{F}) \to \check{H}^1(X, \mathscr{F}'') \end{aligned}$$

is exact. (But note that the sequence does not continue after $\check{H}^1(X, \mathscr{F}'')$.)

Proof. This can be checked "directly". For instance, to construct the connecting homomorphism δ, take an element $s \in \Gamma(X, \mathscr{F}'')$. Locally on X, we can lift it to sections of $\mathscr{F}$, so we obtain an element $(s_i)_i \in C^1(\mathscr{U}, \mathscr{F})$. Its image $(s_{ij})_{i,j}$ in $C^2(\mathscr{U}, \mathscr{F})$ will usually be different from 0 (in fact it is $= 0$ if and only if s is in the image of $\Gamma(X, \mathscr{F})$), but has image 0 in $C^2(\mathscr{U}, \mathscr{F}'')$, and hence comes from an element $(t_{ij})_{i,j} \in C^2(\mathscr{U}, \mathscr{F}')$. Then $(t_{ij})_{i,j}$ has image 0 in $C^3(\mathscr{U}, \mathscr{F}')$ (because that is obviously true in $C^3(\mathscr{U}, \mathscr{F})$ and the morphism $\mathscr{F}' \to \mathscr{F}$ is injective), and so gives rise to a class in $\check{H}^1(\mathscr{U}, \mathscr{F}')$. Its image in $\check{H}^1(X, \mathscr{F}')$ is the image of s under δ. One checks that this procedure is independent of choices and gives rise to the exact sequence in the proposition. □

(6.13) Comparison of cohomology and Čech cohomology.

In degrees 0 and 1, cohomology and Čech cohomology coincide. For degree 0, we have already shown this, so we proceed to the case of degree 1. We start with some preparations.

We define a sheaf version of the Čech complex as follows:

$$\mathscr{C}^p(\mathscr{U},\mathscr{F}) = \prod_{\underline{i}=(i_0<\cdots<i_p)} j_{\underline{i},*}(\mathscr{F}_{|U_{\underline{i}}}),$$

with differentials defined by (basically) the same formula as above. Here $j_{\underline{i}}$ denotes the inclusion $U_{\underline{i}} \hookrightarrow X$.

We have a natural map $\mathscr{F} \to \mathscr{C}^0(\mathscr{U},\mathscr{F})$, which on an open V is given by $s \mapsto (s_{|U_i\cap V})_i$.

Proposition 6.34. *The sequence* $0 \to \mathscr{F} \to \mathscr{C}^0(\mathscr{U},\mathscr{F}) \to \mathscr{C}^1(\mathscr{U},\mathscr{F}) \to \cdots$ *is exact.*

Proof. The exactness can be checked on stalks, and one can show that for each point of X, the stalks of the above complex form a complex that is homotopy equivalent to 0. We omit the details. $\square$

Proposition 6.35. *If $\mathscr{F}$ is flasque, then all $\mathscr{C}^p(\mathscr{U},\mathscr{F})$ are flasque, and $\check{H}^p(\mathscr{U},\mathscr{F}) = 0$ for all $p > 0$.*

Proof. It is not hard to check that the sheaves $\mathscr{C}^p(\mathscr{U},\mathscr{F})$ are flasque, since all the constructions involved preserve flasqueness.

We then obtain

$$\check{H}^p(\mathscr{U},\mathscr{F}) = H^p(\Gamma(\mathscr{C}^\bullet(\mathscr{U},\mathscr{F}))) = H^p(X,\mathscr{F}) = 0,$$

where the second equality holds since $\mathscr{C}^\bullet(\mathscr{U},\mathscr{F})$ is a flasque resolution of $\mathscr{F}$ by the above, and the third one follows since $\mathscr{F}$ itself is flasque. $\square$

Proposition 6.36. *Let X be a topological space and let $\mathscr{F}$ be an abelian sheaf on X.*

(1) *Let $\mathscr{U}$ be an open cover of X. For every i, there is a natural map $\check{H}^i(\mathscr{U},\mathscr{F}) \to H^i(X,\mathscr{F})$.*
(2) *These maps are compatible with refinements, so we obtain a natural map $\check{H}^i(X,\mathscr{F}) \to H^i(X,\mathscr{F})$. These maps are functorial in $\mathscr{F}$.*
(3) *For $i = 0, 1$, the natural map $\check{H}^i(X,\mathscr{F}) \to H^i(X,\mathscr{F})$ is an isomorphism.*

Proof. Part (1) follows from Proposition 6.17 applied to the resolution $\mathscr{C}^\bullet(\mathscr{U},\mathscr{F})$ of $\mathscr{F}$ and any injective resolution. We omit the proof of Part (2).

For Part (3), it remains to consider the case $i = 1$. Embed $\mathscr{F}$ into a flasque sheaf $\mathscr{G}$. We obtain short exact sequences

$$\Gamma(X,\mathscr{G}) \to \Gamma(X,\mathscr{G}/\mathscr{F}) \to \check{H}^1(X,\mathscr{F}) \to 0$$

by Proposition 6.33 Proposition 6.35 and

$$\Gamma(X,\mathscr{G}) \to \Gamma(X,\mathscr{G}/\mathscr{F}) \to H^1(X,\mathscr{F}) \to 0$$

since flasque sheaves are $\Gamma(X,-)$-acyclic. The statement follows from this. □

One can also show that the natural map $\check{H}^2(X,\mathscr{F}) \to H^2(X,\mathscr{F})$ is always injective.

The following result will allow us to compute cohomology groups of separated schemes with coefficients in quasi-coherent modules as Čech cohomology.

Theorem 6.37. (Cartan's Theorem) *Let X be a ringed space, and let $\mathscr{B}$ be a basis of the topology of X which is stable under finite intersections. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Assume that $\check{H}^i(U,\mathscr{F}) = 0$ for all $U \in \mathscr{B}$ and $i > 0$. Then*

(1) *we have $H^i(U,\mathscr{F}) = 0$ for all $U \in \mathscr{B}$ and $i > 0$,*
(2) *The natural homomorphisms $\check{H}^i(\mathscr{U},\mathscr{F}) \to H^i(X,\mathscr{F})$ are isomorphisms for all $i \geq 0$ and all covers $\mathscr{U}$ of X consisting of elements of $\mathscr{B}$.*
(3) *The natural homomorphisms $\check{H}^i(X,\mathscr{F}) \to H^i(X,\mathscr{F})$ are isomorphisms for all $i \geq 0$.*

See e.g., [Go] II Thm. 5.9.2, [Stacks] `01EO` or [GW2] Section (21.17).

Cohomology of affine schemes.

General references: [H] Ch. III, [Stacks], [GW2] Chapter 22.

(6.14) Vanishing of cohomology of quasi-coherent sheaves on affine schemes.

Theorem 6.38. *Let X be an affine scheme, and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then $\check{H}^i(X,\mathscr{F}) = 0$ for all $i > 0$.*

Proof. It is enough to show that $\check{H}^i(\mathscr{U},\mathscr{F}) = 0$ for all covers $\mathscr{U}$ of X by principal open subsets. This can be proved using "direct computation" (see [GW1] Lemma 12.33), or can be viewed as a consequence of the theory of "faithfully flat descent" (specifically Problem 37, see also [GW1] Lemma 14.64). □

From this theorem, it follows immediately (using the above results) that $H^1(X,\mathscr{F}) = 0$ for X affine and $\mathscr{F}$ quasi-coherent. In particular, the global section functor on X preserves exactness of every short exact sequence where the left hand term is a quasi-coherent $\mathscr{O}_X$-module. But using Cartan's Theorem, Theorem 6.37, we get more:

Theorem 6.39. *Let X be an affine scheme, and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then $H^i(X,\mathscr{F}) = 0$ for all $i > 0$.*

Theorem 6.40. *Let X be a separated scheme, and let $\mathscr{U}$ be a cover of X by affine open subschemes. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then the natural homomorphisms $\check{H}^i(\mathscr{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$ are isomorphisms for all $i \geq 0$.*

Remark 6.41. For X noetherian, the use of Cartan's Theorem can be avoided by using the result (see [H] III.3) that for a noetherian ring A, $X = \operatorname{Spec} A$, and I an injective A-module, the $\mathscr{O}_X$-module $\widetilde{I}$ is a flasque $\mathscr{O}_X$-module.

Together with the fact that for an affine scheme X the global section functor is exact on the category of quasi-coherent $\mathscr{O}_X$-modules, this gives the vanishing of $H^i(X, \mathscr{F})$ for X affine, $\mathscr{F}$ quasi-coherent and $i > 0$.

It also implies that any quasi-coherent $\mathscr{O}_X$-module on a noetherian scheme can be embedded into a flasque *quasi-coherent* sheaf. From this one can prove Theorem 6.40.

See [H] III.3, Theorem III.4.5; cf. also [GW2] Section (22.18).

Corollary 6.42. *Let X be a separated scheme which can be covered by $n+1$ affine open subschemes. Then $H^i(X, \mathscr{F}) = 0$ for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every $i > n$.*

Lemma 6.43. *Let X be a scheme. For $f \in \Gamma(X, \mathscr{O}_X)$ write*

$$X_f = \{x \in X;\ f(x) \neq 0 \in \kappa(x)\},$$

an open subset of X which we consider as an open subscheme.

If there exist $f_1, \dots, f_n \in \Gamma(X, \mathscr{O}_X)$ such that X_{f_i} is affine for $i = 1, \dots, n$ and such that $f_1, \dots, f_n$ generate the unit ideal in the ring $\Gamma(X, \mathscr{O}_X)$, then X is affine.

Theorem 6.44. (Serre's criterion for affineness) *Let X be a quasi-compact scheme. The following are equivalent:*

(i) *The scheme X is affine.*

(ii) *For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every $i > 0$, $H^i(X, \mathscr{F}) = 0$.*

(iii) *For every quasi-coherent ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_X$, $H^1(X, \mathscr{I}) = 0$.*

Proof. See [H] Theorem III.3.7 or [GW1] Theorem 12.35. □

Cohomology of projective schemes.

(6.15) The cohomology of line bundles on projective space.

References: [H] III.5, [GW2] Section (22.6), [Stacks] 01XS.

Using Čech cohomology, we can compute the cohomology of line bundles on projective space. It is best to aggregate the results for all $\mathscr{O}(d)$, as we have already seen for their global sections, a result which we repeat as the first statement below.

Theorem 6.45. *Let A be a ring, $n \geq 1$, $S = A[T_0, \dots, T_n]$, $X = \mathbb{P}^n_A$. Then*
(1) *There is a natural isomorphism $S \cong \bigoplus_{d \in \mathbb{Z}} H^0(X, \mathscr{O}(d))$.*
(2) *For $i \neq 0, n$ and all $d \in \mathbb{Z}$ we have $H^i(X, \mathscr{O}(d)) = 0$.*
(3) *There is a natural isomorphism $H^n(X, \mathscr{O}(-n-1)) \cong A$.*
(4) *For every d, there is a perfect pairing*

$$H^0(X, \mathscr{O}(d)) \times H^n(X, \mathscr{O}(-d-n-1)) \to H^n(X, \mathscr{O}(-n-1)) \cong A,$$

i.e., a bilinear map which induces isomorphisms

$$H^0(X, \mathscr{O}(d)) \cong H^n(X, \mathscr{O}(-d-n-1))^\vee$$

and

$$H^0(X, \mathscr{O}(d))^\vee \cong H^n(X, \mathscr{O}(-d-n-1))$$

(where $-^\vee = \mathrm{Hom}_A(-, A)$ denotes the A-module dual).

Sketch of proof. We compute the cohomology groups as Čech cohomology groups for the standard cover $\mathscr{U} = (D_+(T_i))_i$ of $\mathbb{P}^n_A$. It simplifies the reasoning to do the computation for all $\mathscr{O}(d)$ at once, i.e., to compute the cohomology groups of $\mathscr{F} := \bigoplus_{d \in \mathbb{Z}} \mathscr{O}(d)$ (and to – implicitly – keep track of the grading by d). Since cohomology is compatible with direct sums (cf. Proposition 6.29; we proved that for noetherian schemes, but it holds more generally for quasi-compact separated schemes, cf. [GW2] Corollary 21.56), this also gives the result for the individual $\mathscr{O}(d)$.

The Čech complex $C^\bullet(\mathscr{U}, \mathscr{F})$ is

$$0 \to \prod_i S_{T_i} \to \prod_{i,j} S_{T_i T_j} \to \cdots \to S_{T_0 \cdots T_n} \to 0$$

(with non-zero entries in degrees $0, \dots, n$), cf. the computation of the global sections of the sheaves $\mathscr{O}(d)$, Proposition 3.19. Also note that that proposition proves Part (1) of the theorem here. From this, we see that we can identify

$$H^n(\mathbb{P}^n_A, \mathscr{F}) = \bigoplus_{i_0, \dots, i_n < 0} A \cdot T_0^{i_0} \cdots T_n^{i_n} \qquad \subseteq S[T_0^{-1}, \dots, T_n^{-1}].$$

This easily implies Parts (3) and (4).

It remains to prove the vanishing statement of Part (2) for $0 < i < n$. We do this by induction on n. Note that in view of Part (1), all the cohomology groups $H^i(\mathbb{P}^n_A, \mathscr{F})$ carry a natural S-module structure.

First note that for the localization we have $H^i(\mathbb{P}^n_A, \mathscr{F})_{T_n} = 0$. In fact, this localization is the i-th cohomology group of the localized Čech complex $C^\bullet(\mathscr{U}, \mathscr{F})_{T_n}$ which computes the cohomology $H^\bullet(D_+(T_n), \mathscr{F}_{|D_+(T_n)})$ which vanishes in positive degrees. Therefore it suffices to show that multiplication by T_n is a bijection $H^i(\mathbb{P}^n_A, \mathscr{F}) \to H^i(\mathbb{P}^n_A, \mathscr{F})$ for all $0 < i < n$.

The global section $T_n \in H^0(\mathbb{P}^n_A, \mathscr{O}(1))$ gives rise to a short exact sequence

$$0 \to \mathscr{O}(-1) \to \mathscr{O}_{\mathbb{P}^n_A} \to \mathscr{O}_{V_+(T_n)} \to 0,$$

and tensoring with the locally free module $\mathscr{F}$ we obtain a short exact sequence

$$0 \to \mathscr{F} \otimes \mathscr{O}(-1) \to \mathscr{F} \to \mathscr{F}' \to 0$$

where $\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^n_A}} \mathscr{O}_{V_+(T_n)}$ is (the push-forward from $V_+(T_n) \cong \mathbb{P}^{n-1}_A$ to $\mathbb{P}^n_A$) of the sheaf analogous to $\mathscr{F}$ on $\mathbb{P}^{n-1}_A$.

Multiplication by $T_n \in \Gamma(\mathbb{P}^n_A, \mathscr{O}(1))$ gives an isomorphism $\mathscr{F} \otimes \mathscr{O}(-1) \to \mathscr{F}$. Therefore the long exact cohomology sequence attached to the above short exact sequence can be written as

$$\cdots \to H^i(\mathbb{P}^n_A, \mathscr{F}) \to H^i(\mathbb{P}^n_A, \mathscr{F}) \to H^i(\mathbb{P}^{n-1}_A, \mathscr{F}') \to \cdots$$

where the maps $H^i(\mathbb{P}^n_A, \mathscr{F}) \to H^i(\mathbb{P}^n_A, \mathscr{F})$ are given by multiplication by T_n (i.e., what we want to show is that these maps are isomorphisms). The induction hypothesis together with Lemma 6.28 and the observations that the map $H^0(\mathbb{P}^n_A, \mathscr{F}) \to H^0(\mathbb{P}^{n-1}_A, \mathscr{F}')$ is surjective (cf. Part (1)) and that the map $H^{n-1}(\mathbb{P}^{n-1}_A, \mathscr{F}') \to H^n(\mathbb{P}^n_A, \mathscr{F})$ is injective, then allow us to conclude. For the injectivity cf. the above proof for Part (3). Looking at individual graded pieces of $H^{n-1}(\mathbb{P}^{n-1}_A, \mathscr{F}')$ and the kernel of $H^n(\mathbb{P}^n_A, \mathscr{F}) \to H^n(\mathbb{P}^n_A, \mathscr{F})$, we have a surjective homomorphism of free A-modules of the same rank which is necessarily an isomorphism. (The kernel of $H^n(\mathbb{P}^n_A, \mathscr{F}) \to H^n(\mathbb{P}^n_A, \mathscr{F})$ is the free A-module spanned by all monomials of the form $T_0^{i_0} \cdots T_{n-1}^{i_{n-1}} T_n^{-1}$ with all $i_\nu < 0$.) □

Remark 6.46. The homomorphism $H^{n-1}(\mathbb{P}^{n-1}_A, \mathscr{F}') \to H^n(\mathbb{P}^n_A, \mathscr{F})$ at the end of the proof of the previous theorem is given by mapping $T_0^{i_0} \cdots T_{n-1}^{i_{n-1}}$ to $T_0^{i_0} \cdots T_{n-1}^{i_{n-1}} T_n^{-1}$. To verify this, one can use that the long exact cohomology sequence for the short exact sequence $0 \to \mathscr{F} \otimes \mathscr{O}(-1) \to \mathscr{F} \to \mathscr{F}' \to 0$ can be computed in terms of Čech cohomology, namely as the long exact sequence attached to the (exact!) sequence $0 \to C^\bullet(\mathscr{U}, \mathscr{F} \otimes \mathscr{O}(-1)) \to C^\bullet(\mathscr{U}, \mathscr{F}) \to C^\bullet(\mathscr{U}, \mathscr{F}') \to 0$.

(6.16) Finiteness of cohomology of coherent $\mathscr{O}_X$-modules on projective schemes.

References: [H] III.5; [GW2] Sections (23.1), (23.2).

Definition 6.47. *Let X be a noetherian scheme. An $\mathscr{O}_X$-module $\mathscr{F}$ is called* coherent*, if it is quasi-coherent and of finite type.*

Let A be a noetherian ring. For an $\mathscr{O}_{\mathbb{P}^n_A}$-module $\mathscr{F}$, we write $\mathscr{F}(d) := \mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^n_A}} \mathscr{O}(d)$. We need the following general lemma.

Lemma 6.48. *Let X be a quasi-compact and separated scheme, let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and let $s \in \Gamma(X, \mathscr{L})$ be a global section. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module.*

(1) *Let $t \in \Gamma(X, \mathscr{F})$ be a global section such that $t_{|X_s} = 0$. Then there exists an integer $n > 0$ such that $t \otimes s^{\otimes n} = 0 \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$.*
(2) *For every section $t' \in \Gamma(X_s, \mathscr{F})$ there exist $n > 0$ and a section $t \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$ such that $t_{|X_s} = t' \otimes s^{\otimes n}$.*

Proof. If X is affine and $\mathscr{L} = \mathscr{O}_X$, then this follows immediately from our results on quasi-coherent $\mathscr{O}_X$-modules; namely we know that then $\Gamma(X_s, \mathscr{F}) = \Gamma(X, \mathscr{F})_s$. For the general case, let $X = \bigcup_i U_i$ be a finite affine open cover such that $\mathscr{L}_{|U_i} \cong \mathscr{O}_{U_i}$ for all i (and fix such isomorphisms). Then (1) can be checked on each U_i individually and thus follows from what was said in the beginning. To prove Part (2), we construct t by considering sections t_i on the U_i obtained from the restrictions $t_{|U_i \cap X_s}$, using the result in the affine case. The t_i may not glue, but applying Part (1) to the intersections $U_i \cap U_j$ and the elements $t_{i|U_i \cap U_j} - t_{j|U_i \cap U_j}$ we find that for n sufficiently large, the $t_i \otimes s^{\otimes n}$ will glue to a section of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$. See [GW1] Theorem 7.22 or [H] Lemma II.5.14 for more details. □

Proposition 6.49. *Let A be a noetherian ring, $n \geq 1$, let $X = \mathbb{P}^n_A$, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module.*

(1) *There exist integers $d_1, \dots, d_s$ and a surjective $\mathscr{O}_X$-module homomorphism*

$$\bigoplus_{i=1}^{n} \mathscr{O}(d_i) \twoheadrightarrow \mathscr{F}.$$

(2) *For d sufficiently large, the $\mathscr{O}_{\mathbb{P}^n_A}$-module $\mathscr{F}(d)$ is globally generated, i.e., there exist $N \geq 0$ and a surjective homomorphism $\mathscr{O}_{\mathbb{P}^n_A}^N \to \mathscr{F}(d)$.*

Sketch of proof. It is easy to see that (1) and (2) are equivalent. We prove (2). For $i \in \{0, \dots, n\}$, $\mathscr{F}_{|D_+(T_i)}$ has the form $\widetilde{M_i}$ for some finitely generated $A[\frac{T_0}{T_i}, \cdots, \frac{T_n}{T_i}]$-module M_i. For any $s \in M_i = \Gamma(D_+(T_i), \mathscr{F})$, for d sufficiently large, $X_i^d s$ extends to a global section of $\mathscr{F}(d)$, by Lemma 6.48. This implies the claim. □

Theorem 6.50. *Let A be a noetherian ring, X be a projective A-scheme, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then for all $i \geq 0$, the A-module $H^i(X, \mathscr{F})$ is finitely generated.*

Sketch of proof. Using Lemma 6.28, we reduce to the case that $X = \mathbb{P}^n_A$. By Proposition 6.49, the vanishing result Corollary 6.42 and descending induction, we may further reduce to the case that $\mathscr{F}$ is a finite direct sum of sheaves of the form $\mathscr{O}(d)$, but for these we already know the result. □

At this point it is not hard to prove that higher derived images $R^i f_* \mathscr{F}$ of a coherent $\mathscr{O}_X$-module under a projective morphism $f\colon X \to Y$ are coherent (see [H] III.8).

Another useful result is the following vanishing statement.

Proposition 6.51. *Let A be a noetherian ring, $\iota\colon X \to \mathbb{P}^n_A$ a closed immersion, $\mathscr{L} = \iota^* \mathscr{O}_{\mathbb{P}^n_A}(1)$, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. For $n \in \mathbb{Z}$ we write $\mathscr{F}(n) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$. Then there exists $n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$ and all $i > 0$, we have $H^i(X, \mathscr{F}(n)) = 0$.*

Sketch of proof. Using the *projection formula* (see below), one reduces to the case $X = \mathbb{P}^n_A$. It is then clear from the above, that the statement holds whenever $\mathscr{F}$ is a direct sum of line bundles $\mathscr{O}(d)$. The general statement follows from this by descending induction, similarly as above. □

Proposition 6.52. (Projection formula) *Let $f\colon X \to Y$ be a morphism of schemes, let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and let $\mathscr{G}$ be an $\mathscr{O}_Y$-module. There is a natural homomorphism*

$$(f_* \mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G} \to f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^* \mathscr{G})$$

which is an isomorphism if one of the following conditions is satisfied:

(1) *$\mathscr{G}$ is a locally free $\mathscr{O}_Y$-module,*
(2) *f is a closed immersion,*
(3) *$\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent, and for every affine open $V \subseteq Y$, $f^{-1}(V)$ is affine,*
(4) *$\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent, f is quasi-compact and separated, and $\mathscr{G}$ is a flat $\mathscr{O}_Y$-module.*

Sketch of proof. The homomorphism is obtained formally using the adjunction between f^* and f_*. The statement that it is an isomorphism is local on Y. Thus for (1), we may assume that $\mathscr{G}$ is free, and then the claim is easy to check. In situation (2) one can check that the homomorphism induces an isomorphism on each stalk, and hence is an isomorphism. Under the assumptions in (3), one may assume that Y and X are affine and the claim then follows easily from the description of pushforward and pullback in this case (Proposition 2.15). For further details, and for an argument to prove the statement in case (4), see [GW2] Proposition 22.80. □

Serre duality.

(6.17) The Theorem of Riemann–Roch revisited.

References: [H] III.7, IV.1; [GW2] Chapters 25, 26.

Recall the Theorem of Riemann–Roch that we stated above (Theorem 3.13). In this section, we prove a preliminary version, which also gives a more conceptual view on the "error term" $\dim \Gamma(X, \mathscr{O}(K-D))$ (with notation as above).

Let k be an algebraically closed field.

Definition 6.53. *Let X be a projective k-scheme, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. We call*

$$\chi(\mathscr{F}) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathscr{F})$$

the Euler characteristic *of $\mathscr{F}$.*

Note that the sum is finite (by the Grothendieck vanishing theorem, Theorem 6.30) and that each term is finite by the results of the previous section.

Now let X/k be a smooth, projective, connected curve. Then $\chi(\mathscr{F}) = \dim_k H^0(X, \mathscr{F}) - \dim_k H^1(X, \mathscr{F})$.

The following theorem is the preliminary version of the Theorem of Riemann–Roch mentioned above.

Theorem 6.54. *Let $\mathscr{L}$ be a line bundle on X. Then*

$$\chi(\mathscr{L}) = \deg(\mathscr{L}) + \chi(\mathscr{O}_X).$$

Sketch of proof. The statement is clear for $\mathscr{L} = \mathscr{O}_X$. Since every line bundle is isomorphic to the line bundle attached to a Weil divisor, it is therefore enough to prove that for every closed point $x \in X$, and every line bundle $\mathscr{L}$ on X, we have

$$\chi(\mathscr{L}) = \chi(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{O}_X(-[x])) + 1$$

We write $\mathscr{L}(D) := \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{O}_X(D)$ for any divisor D. The short exact sequence

$$0 \to \mathscr{O}_X(-[x]) \to \mathscr{O}_X \to \kappa(x) \to 0$$

remains exact after tensoring with $\mathscr{L}$, so we obtain a short exact sequence

$$0 \to \mathscr{L}(-[x]) \to \mathscr{L} \to \kappa(x) \to 0.$$

Since the Euler characteristic is additive in short exact sequences (use the long exact cohomology sequence) and since $\chi(\kappa(x)) = 1$, the claim follows. $\square$

Now we can define the genus of X as $g := 1 - \chi(\mathscr{O}_X) = \dim_k H^1(X, \mathscr{O}_X)$. From the above, we immediately get

Corollary 6.55. (Theorem of Riemann) *Let $\mathscr{L}$ be a line bundle on X. Then*

$$\dim_k H^0(X, \mathscr{L}) \geq \deg(\mathscr{L}) + 1 - g.$$

Furthermore, the Theorem of Riemann–Roch, Theorem 3.13, follows from the above result on Euler characteristics and the Serre duality theorem (which however we cannot prove in this class).

Theorem 6.56. (Serre duality) *Let k be an algebraically closed field and let X be a connected smooth proper k-scheme. (At this point we can take* smooth *to mean that all local rings $\mathscr{O}_{X,x}$ are regular; in fact it is enough to assume that X is Cohen-Macaulay, i.e., that all local rings of X are Cohen-Macaulay rings.) There is a unique (up to isomorphism) coherent $\mathscr{O}_X$-module ω, the so-called* dualizing sheaf*, such that for every locally free $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank on X, there is a natural isomorphism*

$$H^{n-i}(X, \mathscr{E}) \cong H^i(X, \mathscr{E}^{-1} \otimes \omega)^\vee$$

of k-vector spaces (where $-^\vee$ denotes the dual k-vector space).

If X is smooth over k, then the dualizing sheaf is a line bundle and coincides with the so-called *canonical bundle*, the top exterior power of the sheaf $\Omega^1_{X/k}$ of differentials of X over k, see Chapter 4.

A fairly elementary approach in the case where X is projective is to prove a similar duality theorem for the cohomology of line bundles on projective space $\mathbb{P}^n_k$ (this we have basically done above, Theorem 6.45 (4), with $\omega = \mathscr{O}(-n-1)$), and then to derive the statement for (certain) closed subschemes of projective space; see [H] III.7. A more general (but technically more sophisticated) approach is to derive this duality result from the existence of a right adjoint functor $f^\times$ of the derived push-forward functor $Rf_*\colon D_{\text{qcoh}}(X) \to D_{\text{qcoh}}(\operatorname{Spec} k)$, where $f\colon X \to \operatorname{Spec} k$ is the structure morphism and $D_{\text{qcoh}}(X)$ denotes the full triangulated subcategory of the derived category $D(X)$ of the category of $\mathscr{O}_X$-modules (and this generalizes to the case of arbitrary proper morphisms f between noetherian schemes). In fact, one shows that for X Cohen-Macaulay and equidimensional of dimension n the complex $f^\times k$ is concentrated in degree $-n$, and we denote by ω the unique non-vanishing cohomology object. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and integer i one then has

$$\begin{aligned} H^i(X, \mathscr{F})^\vee &= \operatorname{Hom}_k(Rf_*\mathscr{F}[i], k) = \operatorname{Hom}_{D(X)}(\mathscr{F}[i], \omega[n]) \\ &= \operatorname{Ext}^{n-i}_{\mathscr{O}_X}(\mathscr{F}, \omega), \end{aligned}$$

and if $\mathscr{F}$ is locally free of finite rank, then we can identify the right hand side with

$$\operatorname{Ext}^{n-i}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F}^\vee \otimes_{\mathscr{O}_X} \omega) = H^{n-i}(X, \mathscr{F}^\vee \otimes_{\mathscr{O}_X} \omega).$$

This approach also gives a result (in terms of the derived category) without requiring that X be Cohen-Macaulay, and also for (proper) morphisms $f\colon X \to Y$ where Y is not necessarily the spectrum of a field. See [GW2] Chapter 25 for more on this and for further references.

As another application of Serre duality, we mention the following result. The first part is named the Lemma of Enriques-Severi-Zariski. To name an application, we note that this is one of the ingredients of the proof that every regular proper surface over a field is projective (see [GW2] Theorem 25.151).

The result still holds (but in Part (1) only for H^1, as stated, while for X Cohen-Macaulay one has an analogous result for all H^i, $i < n$) if the assumption that X be Cohen-Macaulay is replaced by the assumption that X is normal (i.e., for every non-empty affine open $U \subseteq X$, the ring $\Gamma(U, \mathscr{O}_X)$ is integrally closed in its field of fractions $K(X)$). On the other hand it is clear that the assumption that X has dimension at least 2 cannot be dropped.

Theorem 6.57. *Let K be an (algebraically closed) field, let X be an integral Cohen-Macaulay (e.g., smooth) projective k-scheme and let $\iota\colon X \to \mathbb{P}^n_k$ be a closed immersion of k-schemes.*

Assume that $\dim X \geq 2$*.*

(1) *Fix $d > 0$ and let $\mathscr{L} := \iota^*\mathscr{O}_{\mathbb{P}^n_k}(d)$. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. We write $\mathscr{F}(n) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$, $n \in \mathbb{Z}$. Then for $n \ll 0$, $H^1(X, \mathscr{F}(n)) = 0$.*
(2) *Let $f \in k[T_0, \dots, T_n]$ be a non-constant homogeneous polynomial, and let $Z := V_+(f)$, a closed subscheme of $\mathbb{P}^n_k$. Then $X \cap Z$ is connected.*

Sketch of proof. Part (1) follows from Serre duality (Theorem 6.56) and the vanishing result of Proposition 6.51. To prove Part (2), let d denote the degree of f. We view Z as an effective Cartier divisor. Then $\mathscr{O}_{\mathbb{P}^n_k}(Z) \cong \mathscr{O}(d)$. We set $\mathscr{L} = \iota^*\mathscr{O}(d)$ and $\mathscr{F} = \mathscr{O}_X$ and apply Part (1) to find n such that $H^1(X, \mathscr{L}^{-n}) = 0$. Let Z_n be the Cartier divisor $n \cdot Z$ (with associated line bundle $\cong \mathscr{O}(dn)$). We then have a short exact sequence

$$0 \to \mathscr{L}^{-n} \to \mathscr{O}_X \to \mathscr{O}_{X\cap Z_n} \to 0,$$

where $X \cap Z_n$ denotes the scheme-theoretic intersection of X and Z, i.e., $\mathscr{O}_{X\cap Z_n} = \iota^*\mathscr{O}_{Z_n}$. The underlying topological space of $X \cap Z_n$ is independent of $n \geq 1$ and equals the set-theoretic intersection $X \cap Z$. It is therefore sufficient to show that the scheme $X \cap Z_n$ is connected. But the above short exact sequence induces, in view of the vanishing of $H^1(X, \mathscr{L}^{-n})$ given by Part (1), a surjective homomorphism $\Gamma(X, \mathscr{O}_X) \to \Gamma(X, \mathscr{O}_{X\cap Z_n})$ of k-vector spaces. Since $\Gamma(X, \mathscr{O}_X) = k$, it follows that $\Gamma(X, \mathscr{O}_{X\cap Z_n}) = k$, as well, and in particular $X \cap Z_n$ is connected. See also [GW2] Section (25.28), [H] Section III.7. □

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