Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2022/23 Dr. Heer Zhao

Problem sheet 9

Due date: Dec. 13, 2022.

Problem 33

Let k be an algebraically closed field of characteristic $\neq 2$. Let $X = D(T+1) \subseteq \mathbb{A}^1_k$ (where T is the coordinate on $\mathbb{A}^1_k$, i.e., $\mathbb{A}^1_k = \operatorname{Spec} k[T]$), and let $Y = V(U^2 - T^2(T+1)) \subseteq A^2_k$ (with coordinates T, U). We view Y as the scheme $\operatorname{Spec} k[T,U]/(U^2 - T^2(T+1))$.

Show that there is a morphism $f\colon X \to Y$ of schemes which on closed points is given as $t \mapsto (t^2-1, t(t^2-1))$.

Show that f is a bijection on the underlying topological spaces, but not an isomorphism of schemes.

Hint. You may make use of Hilbert's Nullstellensatz and of the fact that $\dim X = \dim Y = 1$, i.e., all non-zero prime ideals in the affine coordinate rings of X and Y are maximal.

Problem 34

Let X be a scheme and let $f \in \Gamma(X, \mathscr{O}_X)$. Show that

$$X_f := \{x \in X; f(x) \neq 0 \in \kappa(x)\}$$

is an open subset of X. Show that the image of f in $\Gamma(X_f, \mathscr{O}_X)$ is a unit in this ring.

Problem 35

Let p be a prime number. We say that a ring A *has characteristic* p if $p \cdot 1 = 0$ in A. Let $\iota\colon \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ be the canonical morphism. Let X be a scheme. Prove that the following are equivalent:

(1) The ring $\Gamma(X, \mathscr{O}_X)$ has characteristic p.

(2) For all open subsets $U \subseteq X$, the ring $\Gamma(U, \mathscr{O}_X)$ has characteristic p.

(3) The unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ factors as

$$X \longrightarrow \operatorname{Spec} \mathbb{F}_p \xrightarrow{\iota} \operatorname{Spec} \mathbb{Z}$$

If the conditions are satisfied, we say that X *has characteristic* p. Show that in this case the morphism $X \to \operatorname{Spec} \mathbb{F}_p$ is unique.

Give an example of a scheme X such that all residue class fields of the local rings of X have characteristic p, but such that X does not satisfy the above conditions.

Problem 36

Let X be a scheme of characteristic p. Show that there exists a unique morphism $(F, F^\flat)\colon X \to X$ of schemes such that on topological spaces, $F = \mathrm{id}_X$, and for an open $U \subseteq X$, $F^\flat$ is given by $\Gamma(U, \mathscr{O}_X) \to \Gamma(U, \mathscr{O}_X)$, $s \mapsto s^p$. This morphism is called the *absolute Frobenius morphism* of X.

Give an example where F is not an isomorphism.