Algebraic Geometry I — Prof. Dr. Ulrich Görtz
WS 2022/23 — Dr. Heer Zhao

Problem sheet 5

Due date: Nov. 15, 2022.

Problem 17

Let X be a topological space, and suppose that $X = \bigcup_{i=1}^n Z_i$, where the Z_i, $i = 1, \dots, n$, are closed irreducible subsets of X such that $Z_i \not\subseteq Z_j$ for $i \neq j$. Prove that the Z_i are precisely the irreducible components of X.

Problem 18

Determine the irreducible components of

$$\operatorname{Spec} k[X, Y, Z]/(XY, XZ)$$

(in the form $V(\mathfrak{a})$ for $\mathfrak{a}$ an ideal of $k[X, Y, Z]/(XY, YZ)$).

Problem 19

(1) Let $\psi\colon A \to B$ be an injective ring homomorphism between reduced rings. Show that every minimal prime ideal of A is in the image of ${}^a\psi$. Give an example where ${}^a\psi$ is not surjective.

(2) Let $\varphi\colon A \to B$ be a ring homomorphism, and let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the map attached to φ. Assume that f is bijective and that f reflects specialization, i.e., for $x, x' \in \operatorname{Spec} B$ we have $f(x') \in \overline{\{f(x)\}}$ if *and only if* $x' \in \overline{\{x\}}$. Show that f is a homeomorphism.

Hint. In (2), we need to show that f is closed. Let $\mathfrak{b} \subset B$ be a radical ideal. Apply (1) to the ring homomorphism $A/\varphi^{-1}(\mathfrak{b}) \to B/\mathfrak{b}$ induced by φ and then use that f reflects specialization to show that $f(V(\mathfrak{b})) = V(\varphi^{-1}(\mathfrak{b}))$.

Problem 20

Give an example of a continuous map $f\colon X \to Y$ between topological spaces X and Y which is bijective and compatible with specialization (i.e., for $x, x' \in X$ we have $x' \in \overline{\{x\}}$, if and only if $f(x') \in \overline{\{f(x)\}}$) which is not a homeomorphism.