Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2022/23 Dr. Heer Zhao

Problem sheet 4

Due date: Nov. 08, 2022.

Problem 13

Let k be an algebraically closed field, let $m, n \geqslant 0$, and let $f_1, \cdots, f_m \in k[T_1, \cdots, T_n]$. We consider k^m and k^n as topological spaces with respect to the Zariski topology. Let θ be the map

$$k^n \longrightarrow k^m, \quad (t_1, \ldots, t_n) \mapsto (f_1(t_1, \ldots, t_n), \ldots, f_m(t_1, \ldots, t_n)),$$

and let φ be the homomorphism of k-algebras

$$k[T_1, \cdots, T_m] \to k[T_1, \cdots, T_n], \quad T_i \mapsto f_i.$$

We regard k^n as a subset of $\operatorname{Spec} k[T_1, \cdots, T_n]$ via the bijection

$$k^n \to \operatorname{Max}(k[T_1, \cdots, T_n])$$

given by Hilbert's Nullstellensatz[1]. Similarly we regard k^m as a subset of $\operatorname{Spec} k[T_1, \cdots, T_m]$. Show that the following diagram

$$\begin{array}{ccc} k^n & \xrightarrow{\quad\theta\quad} & k^m \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} k[T_1, \cdots, T_n] & \xrightarrow{\varphi^{\mathrm{a}}} & \operatorname{Spec} k[T_1, \cdots, T_m] \end{array}$$

is commutative.

Problem 14

Let k be an algebraically closed field. We consider k^n as a topological space with respect to the Zariski topology. Give an example of a closed subset of k^2 whose image under the projection $k^2 \to k$, $(a, b) \mapsto a$ is not closed.

Hint. One possibility is to find a suitable polynomial $f \in k[X, Y]$ such that $f(0, b) \neq 0$ for any $b \in k$.

Problem 15

Let A be a ring. We call an element $e \in A$ *idempotent* if $e^2 = e$. Show that the following conditions are equivalent:

(i) $\operatorname{Spec} A$ is not connected.

[1] “Nullstellen” is “zeros” in German, and “Satz” is “theorem” in German.

(ii) There exists an idempotent element $e \in A$ different from 0 and 1.

(iii) There exists a ring isomorphism $A \cong A_1 \times A_2$ with non-zero rings A_1, A_2.

Problem 16

Let A be a ring, $f \in A$, M an A-module. Consider the inductive system of A-modules (with index set $\mathbb{Z}_{\geqslant 0}$)

$$M \to M \to M \to \dots$$

where all transition maps are given by multiplication by f. Show that there exists a natural isomorphism between the colimit $\operatorname{colim}_i M$ and the localization M_f of the A-module M with respect to the element f.