

**Problem sheet 13**

Due date: Jan. 24, 2022.

**Problem 49**

- (1) Let  $X$  be an irreducible noetherian scheme with generic point  $\eta$  such that the local ring  $\mathcal{O}_{X,\eta}$  is reduced. Show that there exists a non-empty open subscheme  $U \subseteq X$  which is reduced.
- (2) Give an example of a non-reduced irreducible noetherian scheme  $X$  such that the local ring at the generic point is reduced.

**Problem 50**

Show that open and closed immersions are *monomorphisms* in the category of schemes: If  $f: X \rightarrow Y$  is an open (closed) immersion, then for every scheme  $S$  the induced map  $\text{Hom}(S, X) \rightarrow \text{Hom}(S, Y)$  is injective.

**Problem 51**

- (1) Let  $X$  be a scheme. Prove that there exists a unique reduced closed subscheme  $X_{\text{red}}$  of  $X$  which has the same underlying topological space as  $X$ .
- (2) Let  $f: X \rightarrow Y$  be a morphism of schemes. Show that  $f$  induces a unique morphism  $f_{\text{red}}: X_{\text{red}} \rightarrow Y_{\text{red}}$  such that the diagram

$$\begin{array}{ccc} X_{\text{red}} & \hookrightarrow & X \\ f_{\text{red}} \downarrow & & \downarrow f \\ Y_{\text{red}} & \hookrightarrow & Y \end{array}$$

is commutative.

**Problem 52**

Let  $k$  be a field, and let  $A = k[X, Y]/(XY, X^2)$ . Define two morphisms

$$f, g: \text{Spec } A \rightarrow \text{Spec } k[T]/(T^2)$$

such that  $f \neq g$ , but such that there exists a non-empty open subset  $U \subset \text{Spec } A$  such that  $f|_U = g|_U$ .