Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2022/23 Dr. Heer Zhao

Problem sheet 10

Due date: Dec. 20, 2022.

Problem 37

Let k be an algebraically closed field, $Z = V(T_1, \dots, T_n) \subset \mathbb{A}_k^n$. Determine for which $n \geqslant 1$ the open subscheme $X := \mathbb{A}_k^n \smallsetminus Z$ is affine.

Problem 38

Let k be an infinite field, let $n \geqslant 1$, and let $U \subseteq \mathbb{A}_k^n$ be a non-empty open subscheme. Show that there exists a morphism $\operatorname{Spec} k \to U$ of k-schemes (i.e., that U has a k-valued point). *Hint.* Problem 2 (1).

Give an example of an infinite field k and a non-empty affine k-scheme X which has no k-valued points.

Problem 39

We call a ring A *reduced*, if $0 \in A$ is the only nilpotent element of A. Now let X be a scheme. Prove that the following conditions are equivalent. (If they are satisfied, then we call X a *reduced scheme*.)

(i) For every open subset $U \subseteq X$, the ring $\Gamma(U, \mathscr{O}_X)$ is reduced.

(ii) For every open cover $X = \bigcup_i U_i$ by affine schemes $(U_i, \mathscr{O}_{X|U_i})$, for every i, the ring $\Gamma(U_i, \mathscr{O}_{X|U_i})$ is reduced.

(iii) There exists an open cover $X = \bigcup_i U_i$ by affine schemes $(U_i, \mathscr{O}_{X|U_i})$, such that for every i, the ring $\Gamma(U_i, \mathscr{O}_{X|U_i})$ is reduced.

(iv) For every $x \in X$, the ring $\mathscr{O}_{X,x}$ is reduced.

Problem 40 Let k be an infinite field, let $n \geqslant 1$, and let $f \in k[T_1, T_2, \cdots, T_{n+1}]$. We write $f = \sum_i f_i$ with f_i the degree i homogeneous part of f. Let

$$P = (a_1, a_2, \cdots, a_{n+1}) \in k^{n+1} \smallsetminus \{(0, 0, \cdots, 0)\}$$

be such that $f(\lambda a_1, \lambda a_2, \cdots, \lambda a_{n+1}) = 0$ for any $\lambda \in k^\times$. Show that

$$f_i(a_1, a_2, \cdots, a_{n+1}) = 0$$

for any i.