

## II The spectrum of a ring

26.10.2022

Reference: [GW] (2.1) - (2.4).

Görtz, Wedhorn, Algebraic Geometry I, see Moodle  
page

## II.1 Artin's approach to algebraic geometry in a nutshell

- Classical approach

Hilbert's Nullstellensatz:  $k$  algebraically closed field

$$\text{Max}(k[T_1, \dots, T_n]) \xleftrightarrow{1:1} k^n$$

$$(T_1 - t_1, \dots, T_n - t_n) \longleftarrow (t).$$

set of all  
maximal  
ideals

Given  $f_1, \dots, f_m \in k[T_1, \dots, T_n]$ , the above

bijection restricts to a bijection

$$\text{Max}(k[T_1, \dots, T_n] / (f_1, \dots, f_m)) \xleftrightarrow{1:1} V(f_1, \dots, f_m)$$

Proof: •  $\text{Max}(k[T_1, \dots, T_n] / \mathfrak{a}) \xleftrightarrow{1:1} \left\{ \begin{array}{l} \mathfrak{m} \in \text{Max}(k[T_1, \dots, T_n]) ; \\ \mathfrak{a} \subseteq \mathfrak{m} \end{array} \right\}$

- $f \in (T_1 - t_1, \dots, T_n - t_n) \iff f(t_1, \dots, t_n) = 0,$

in other words,  $(T_1 - t_1, \dots, T_n - t_n) \stackrel{*}{=} \text{Ker } \Phi,$

where  $\Phi: k[T_1, \dots, T_n] \rightarrow k, T_i \mapsto t_i, \circlearrowleft$

the evaluation homomorphism.

For  $\otimes$ , " $\subseteq$ " is clear.

For the inclusion " $\supseteq$ " one can either argue "directly" (e.g. using polynomial division), or use that  $(T_1 - t_1, \dots, T_n - t_n)$  is a maximal ideal.

---

→ try to understand properties of  $V(f_1, \dots, f_n)$  by studying the ring  $k[T_1, \dots, T_n]/(f_1, \dots, f_n)$

### Problems

- situation is more complicated, if  $k$  not algebraically closed
- in general, set of maximal ideals of a ring not well behaved (e.g. for ring homom.  $R \xrightarrow{\varphi} S$ ,  $\mathfrak{m} \subset S$  max id,  $\varphi^{-1}(\mathfrak{m})$  not max in general)

- Another idea: attach, to any commutative ring  $R$ , the set  $\text{Spec } R$  of all prime ideals in  $R$ .

$\text{Spec } R$  becomes a topological space when equipped with Zariski topology. (Algebra 2/ see below)

→ functor  $\underbrace{(\text{Rings})^{\text{op}}}_{\text{cat. of comm. rings}} \rightarrow \text{Top}$  / Category of topological spaces

Turns out: to fully capture the geometric aspects of the situation, one should (and can) add further structure and equip  $\text{Spec } R$  with what is called the structure of a locally ringed space (see below).

Then we will also be able to construct more general objects (such as projective space) by "gluing".

## II.2 Reminds: the prime spectrum of a ring

Def (1) For a ring (always: with unit, commutative)  $A$ , let  $\text{Spec } A := \{ \mathfrak{p} \subset A \text{ prime ideal} \}$

(2) We equip  $\text{Spec } A$  with the Zariski topology whose closed sets, by definition, are the sets of the form

NB. There is a small conflict of notation here with regard to the  $V(\mathfrak{a})$  of the introduction.

$$V(\mathfrak{a}) = \{ \mathfrak{p} \in \text{Spec } A; \mathfrak{a} \subseteq \mathfrak{p} \}, \quad \mathfrak{a} \subseteq A \text{ an ideal.}$$

(3) If  $\varphi: A \rightarrow B$  is a ring homomorphism,

$$\text{then we define } \varphi^a: \text{Spec } B \rightarrow \text{Spec } A \\ \mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q}).$$

Lemma. The map  $\varphi^a$  on Part (3) of the definition

is continuous. We obtain a contravariant

$$\text{functor } (\text{Rings})^{\text{op}} \rightarrow (\text{Top}), \quad \leftarrow \begin{array}{l} \text{category} \\ \text{of topol. spaces} \end{array} \\ A \mapsto \text{Spec } A.$$



Example (0) For a ring  $A$ ,  $\text{Spec} A = \emptyset \Leftrightarrow A = 0$

(1)  $\text{Spec } \mathbb{Z} = \{(0)\} \cup \{(p)\}; p \in \mathbb{Z} \text{ prime number}$

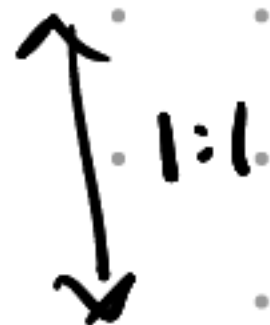
$\text{Spec } k[X] = \{(0)\} \cup \{(f)\}; f \in k[X] \text{ monic, irreducible}$   
 $k$  a field

(similarly for every principal ideal domain).

(2)  $k$  algebraically closed,

$A = k[X_1, \dots, X_n] / (f_1, \dots, f_m)$  set of max'l  
ideals of  $A$

$\text{Spec } A = \text{Max}(A) \cup \underbrace{\{\mathfrak{p} \subset A \text{ prime ideal, not maximal}\}}_{\text{not maximal}}$



"classical  
algebraic  
geometry"

$\Leftrightarrow \{(x_1, \dots, x_n) \in k^n; \forall j: f_j(x_1, \dots, x_n) = 0\}$

one of our next  
goals:  
understand this  
part better

