II The spectrum of a ring

26.10.2022

Reference: [GW] (2.1) – (2.4).

Görtz, Wedhorn, Algebraic Geometry I, see Moodle page

II.1 Grothendieck's approach to algebraic geometry in a nutshell

- <u>Classical approach</u>

Hilbert's Nullstellensatz: k alg. closed field

$$\mathrm{Max}\left(k[T_1,\dots,T_n]\right) \xleftrightarrow{1:1} k^n$$

$$(T_1-t_1,\dots,T_n-t_n) \longleftarrow (t_\bullet).$$

set of all maximal ideals

Given $f_1,\dots,f_m \in k[T_1,\dots,T_n]$, the above bijection restricts to a bijection

$$\mathrm{Max}\left(k[T_1,\dots,T_n]/(f_1,\dots,f_m)\right) \xleftrightarrow{1:1} V(f_1,\dots,f_m)$$

Proof. • $\mathrm{Max}\left(k[T_\bullet]/\mathfrak{a}\right) \xleftrightarrow{1:1} \left\{\mathfrak{m} \in \mathrm{Max}(k[T_\bullet]) \; ; \; \mathfrak{a} \subseteq \mathfrak{m}\right\}$

• $f \in (T_1-t_1,\dots,T_n-t_n) \iff f(t_1,\dots,t_n)=0$, in other words, $(T_1-t_1,\dots,T_n-t_n) \overset{\circledast}{=} \mathrm{Ker}\,\Phi$, where $\Phi : k[T_1,\dots,T_n] \to k$, $T_i \mapsto t_i$, is the evaluation homomorphism.

For ⊛, "$\subseteq$" is clear.

For the inclusion "$\supseteq$" one can either argue "directly" (e.g. using polynomial division), or use that $(T_1 - t_1, \dots, T_n - t_n)$ is a maximal ideal.

⟶ try to understand properties of $V(f_1, \dots, f_m)$ by studying the ring $k[T_1, \dots, T_n]/(f_1, \dots, f_m)$

<u>Problems</u>

- situation is more complicated, if k not algebraically closed
- in general, set of maximal ideals of a ring not well behaved (e.g. for ring homom $R \xrightarrow{\varphi} S$, $\mathfrak{n} \subset S$ max'l, $\varphi^{-1}(\mathfrak{n})$ not max in general)

- Grothendieck: attach, to any commutative ring R, the set $\operatorname{Spec} R$ of all <u>prime</u> ideals in R.

Spec R becomes a topological space when equipped with Zariski topology (Algebra 2/ see below)

$\rightsquigarrow$ functor $\underbrace{(\text{Rings})}_{\text{cat. of comm. rings}}^{op} \longrightarrow \text{Top}$

category of topological spaces

Turns out: to fully capture the geometric aspects of the situation, one should (and can) add further structure and equip Spec R with what is called the structure of a locally ringed space (see below).

Then we will also be able to construct more general objects (such as projective space) by "gluing".

II.2 Reminder: the prime spectrum of a ring

Def (1) For a ring (always: with unit, commutative) A, let $\operatorname{Spec} A := \{ \mathfrak{p} \subset A \text{ prime ideal} \}$

(2) We equip $\operatorname{Spec} A$ with the <u>Zariski topology</u> whose closed sets, by definition, are the sets of the form

$$V(\mathfrak{a}) = \{ \mathfrak{p} \in \operatorname{Spec} A ;\ \mathfrak{a} \subseteq \mathfrak{p} \}, \quad \mathfrak{a} \subseteq A \text{ an ideal.}$$

NB. There is a small conflict of notation here with regard to the $V(\mathfrak{a})$ of the introduction.

(3) If $\varphi : A \to B$ is a ring homomorphism, then we define
$$\varphi^a : \operatorname{Spec} B \to \operatorname{Spec} A$$
$$\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q}).$$

<u>Lemma</u>. The map φ^a in Part (3) of the definition is continuous. We obtain a contravariant functor
$$(\mathrm{Rings})^{op} \to (\mathrm{Top}), \quad A \mapsto \operatorname{Spec} A.$$
category of topol. spaces

Example (0) For a ring A, $\operatorname{Spec} A = \emptyset \iff A = 0$

(1) $\operatorname{Spec} \mathbb{Z} = \{(0)\} \cup \{(p) ;\ p \in \mathbb{Z} \text{ prime number}\}$

$\operatorname{Spec} k[X] = \{(0)\} \cup \{(f) ;\ f \in k[X] \text{ monic, irreducible}\}$

k a field

(similarly for every principal ideal domain).

(2) k algebraically closed,

$$A = k[X_1, \dots, X_n] / (f_1, \dots, f_m)$$

set of max'l ideals of A

$$\operatorname{Spec} A = \operatorname{Max}(A) \ \dot\cup\ \underbrace{\{\mathfrak{p} \subset A \text{ prime ideal, not maximal}\}}$$

$\updownarrow$ 1:1

"classical algebraic geometry" $\longleftrightarrow$ $\{(x_\bullet)_\bullet \in k^n ;\ \forall j : f_j(x_\bullet) = 0\}$

one of our next goals: understand this part better