III Sheaves

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reference: [GW] Sections (2.5) ff.

Slogan: "(type of) geometry" is determined by the (type of) functions one considers:

topological spaces $\leftrightarrow$ continuous functions

differentiable manifolds $\leftrightarrow$ differentiable fcts

complex manifolds $\leftrightarrow$ holomorphic functions

(algebraic geometry $\leftrightarrow$ polynomials)

III.1 Presheaves and sheaves

X topological space

Def A presheaf $\mathcal{F}$ (of sets) on X is given by

- for every open $U \subseteq X$ a set $\mathcal{F}(U)$
- for every two open $V \subseteq U \subseteq X$ a map ("restriction map") $\mathcal{F}(U) \xrightarrow{res^U_V} \mathcal{F}(V)$

such that
- $res^U_U = id_{\mathcal{F}(U)}$ for all U
- $res^V_W \circ res^U_V = res^U_W$ for all $W \subseteq V \subseteq U$

(i.o.w.: a functor $Open(X)^{op} \longrightarrow (Sets)$,

where $Open(X)$ is the category with objects the open subsets of X, morphisms
$$Hom(U,V) = \begin{cases} \{*\} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise} \end{cases}$$
(and the unique identity elements and composition))

Def Let $\mathcal{F}, \mathcal{G}$ be presheaves on a topological space X. A <u>morphism</u> $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ of presheaves is a family of maps $\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U)$, $U \subseteq X$ open, s.t. for all $V \subseteq U \subseteq X$ open, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "res_{\mathcal{F}}"'] & \mathcal{G}(U) \arrow[d, "res_{\mathcal{G}}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

commutes.

Notation / Terminology:

- $s|_V := res^U_V(s), \quad s \in \mathcal{F}(U)$
- the elements of $\mathcal{F}(U)$ are called the <u>sections</u> of $\mathcal{F}$ on U

Def A sheaf (of sets) on X is a presheaf $\mathcal{F}$ such that for every $U \subseteq X$ open and every cover $U = \bigcup_{i \in I} U_i$ by open subsets $U_i \subseteq U$,

(a) If $s, s' \in \mathcal{F}(U)$ s.t. $s|_{U_i} = s'|_{U_i}$ $\forall i$, then $s = s'$.

(b) If $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j$, then there exists a (unique) $s \in \mathcal{F}(U)$ s.t. $s_i = s|_{U_i}$ $\forall i$.

In other words:

$$\mathcal{F}(U) \xrightarrow{\rho} \prod_i \mathcal{F}(U_i) \overset{\sigma}{\underset{\sigma'}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

$$s \longmapsto (s|_{U_i})_i \qquad (s_i)_i \longmapsto (s_i|_{U_i \cap U_j})_{i,j}$$
$$\qquad\qquad\qquad\qquad\qquad \longmapsto (s_j|_{U_i \cap U_j})_{i,j}$$

is exact (or: an equalizer), i.e. ρ injective and $\mathrm{im}(\rho) = \{ x \in \prod \mathcal{F}(U_i) \; ; \; \sigma(x) = \sigma'(x) \}$

In particular: $\mathcal{F}(\emptyset) = \{*\}$ (a one-point set) for every sheaf $\mathcal{F}$.

Def Let $\mathcal{F}, \mathcal{G}$ be sheaves on X.

A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves is a morphism $\mathcal{F} \to \mathcal{G}$ of presheaves.

Similarly: sheaves of groups, ab. groups, rings, R-modules (R a ring).

Examples

(1) X, Y topological spaces, then $\mathcal{F}(U) := \{ f : U \to Y \text{ continuous} \}$ (with restriction of functions as restriction maps) defines a sheaf on X.

(2) X differentiable manifold (e.g. $X \subseteq \mathbb{R}^n$ open) $\rightsquigarrow$ $\mathcal{F}(U) := \{ f : U \to \mathbb{R} \text{ differentiable} \}$ defines a sheaf of rings on X.

(3) X complex manifold (e.g. $X \subseteq \mathbb{C}^n$ open)

$\rightsquigarrow$ $\mathcal{F}(U) := \{ f : U \to \mathbb{C} \text{ holomorphic} \}$

defines a sheaf of rings on X

(4) $X = \mathbb{R}$ (with the usual topology),

$\mathcal{F}(U) := \{ f : U \to \mathbb{R} \text{ bounded } (+\text{ differentiable}) \}$

defines a presheaf on X which is not a sheaf.

<u>Rmk.</u> In general, the restriction maps of a sheaf are neither injective nor surjective.

(But there are interesting situations where they are, e.g. for X a complex manifold, $\mathcal{F}$ as in (3) above, $\emptyset \neq V \subseteq U \subseteq X$ open and U connected, the identity theorem says that $\mathcal{F}(U) \hookrightarrow \mathcal{F}(V)$ is injective.)