# IV Schemes

(reference: [GW] (2.9) - (2.12), Chapter 3.

## IV.1 Locally ringed spaces

Have seen: to a ring $A$ we can attach

- a topological space $\operatorname{Spec} A$, and
- a sheaf of rings $\mathcal{O}_A$

Would like to turn this into a functor

$$\operatorname{Spec}: (\text{Rings})^{op} \longrightarrow \left(\begin{array}{c}\text{"topol spaces with} \\ \text{sheaves of rings"}\end{array}\right)$$

such that Spec is fully faithful, i.e. induces bijections

$$\operatorname{Hom}_{(\text{Rings})}(A,B) \to \operatorname{Hom}_{LR}\left((\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}), (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})\right)$$

To achieve this, we need to

- define a suitable notion of morphism on the class of pairs (topol space, sheaf of rings)
- restrict to a suitable class of such pairs in order to ensure there are not too many morphisms on the right hand side

**Def.** A <u>ringed space</u> is a pair $(X, \mathcal{O}_X)$ consisting of a topol space $X$ and a sheaf $\mathcal{O}_X$ of rings on $X$, called the <u>structure sheaf</u> of $X$.

A morphism between ringed spaces $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\flat)$ where $f$ is a continuous map $X \to Y$ and $f^\flat$ is a morphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$ of sheaves of rings on $Y$.

<u>Example</u> $X$ topol space, $X^{top} = (X, \mathcal{O}_X^{top})$ where $\mathcal{O}_X^{top}(U) = \{ U \to \mathbb{R} \text{ continuous} \}$ (with restriction of functions as restriction maps)

For $X \xrightarrow{f} Y$ a continuous map of topological spaces, obtain $\mathcal{O}_Y^{top} \xrightarrow{f^\flat} f_* \mathcal{O}_X^{top}$ by composition

$$\Big( f^{-1}(V) \underset{f|_{f^{-1}(V)}}{\longrightarrow} V \underset{s \in \mathcal{O}_Y^{top}(V)}{\longrightarrow} \mathbb{R} \Big) \in (f_* \mathcal{O}_X^{top})(V), \quad \begin{matrix} V \subseteq Y \\ \text{open} \end{matrix}$$

$\rightsquigarrow$ morphism $X^{top} \to Y^{top}$ of ringed spaces.

Similarly: $X$ differentiable / complex manifold, $\mathcal{O}_X$ sheaf of differentiable fcts $U \to \mathbb{R}$ / holomorphic functions $U \to \mathbb{C}$.

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Given $f: X \to Y$ morphism of ringed spaces (where we, as usual, do not mention $\mathcal{O}_X$, $\mathcal{O}_Y$, $f^\flat$ explicitly), from $f^\flat$ by adjunction we obtain

$$f^\#: f^{-1}\mathcal{O}_Y \to \mathcal{O}_X \quad \left(\begin{matrix}\text{morph of} \\ \text{sheaves on } X\end{matrix}\right)$$

and therefore, for every $x \in X$ a ring homom

$$f_x^\#: \mathcal{O}_{Y, f(x)} = (f^{-1}\mathcal{O}_Y)_x \longrightarrow \mathcal{O}_{X,x}$$

(in the above examples, this is again composition of "germs of functions" (i.e. fcts defined in an open neighborhood of $f(x)$) with $f$).

In the above examples (when the structure sheaf is defined in terms of functions to a field $k$) we have a natural map $\mathcal{O}_{X,x} \xrightarrow{ev_x} k$ by evaluating a function at $x$, which is surjective and such that $\mathcal{O}_{X,x} \setminus \ker(ev_x) = \mathcal{O}_{X,x}^{\times}$.

Therefore the following is a natural definition:

**Def.** A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that for every $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a <u>local</u> ring, i.e., has a unique maximal ideal.

Given locally ringed spaces $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is a morphism $(f, f^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces such that for every $x \in X$ the ring homom. $f_x^\# : \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X,x}$ is a <u>local</u> ring homom., i.e. the image of the max'l ideal of $\mathcal{O}_{Y,f(x)}$ is contained in the max'l ideal of $\mathcal{O}_{X,x}$.

$\boxed{6.12.2022}$

Have defined

(Ringed Sp) category of ringed spaces

(Loc Ringed Sp) category of locally ringed spaces

Remark (local ring homomorphisms)

Let $\varphi: A \to B$ be a ring homomorphism between local rings $A, B$ with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$.

Then $\varphi$ local $:\iff \varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$

$\iff \varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$

$\iff \varphi$ fits into a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A/\mathfrak{m}_A \arrow[r] & B/\mathfrak{m}_B \end{tikzcd},$$

i.e. $\varphi$ induces a homomorphism $\kappa(\mathfrak{m}_A) \longrightarrow \kappa(\mathfrak{m}_B)$ between the residue class fields.