VI Further properties of schemes

VI.1 Topological properties, noetherian schemes

Def (1) A scheme is called

connected / quasi-compact / irreducible

if its underlying topological space has this property.

(2) A morphism of schemes is called

injective / surjective / bijective /

open* / closed** / a homeomorphism,

if the corresponding continuous map has this property.

* $f: X \to Y$ open: $\forall U \subseteq X$ open: $f(U) \subseteq Y$ open

** $f: X \to Y$ closed: $\forall Z \subseteq X$ closed: $f(Z) \subseteq Y$ closed

Def A morphism $f: X \to Y$ of schemes is called

quasi-compact, if for every quasi-compact open subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.

Noetherian schemes

Def (1) A topological space is called <u>noetherian</u> if it satisfies the descending chain condition for closed subsets.

(2) A scheme X is called <u>locally noetherian</u> if it admits an affine open cover $X = \bigcup_{i \in I} U_i$ s.t., for every i, $\Gamma(U_i, \mathcal{O}_X)$ is a noetherian ring.

(3) A scheme X is called <u>noetherian</u> if it is locally noetherian and quasi-compact.

Remark (0) X topol space. Then

X noeth $\iff$ every non-empty set of closed subsets of X has a minimal elt $\iff$ every non-empty set of open subsets of X has a maximal element

(1) X noeth topol space. Then every subspace of X is noetherian.

(2) X topol space. Then

X noetherian $\iff$ every open subset of X is quasi-compact

(3) X noetherian topol space

$\Rightarrow$ X has only finitely many irreducible components.

(4) X noeth scheme $\Rightarrow$ the underlying topol space of X is noetherian

(but $\Leftarrow$ does not hold in general!)

(For proofs, see [GW, Section (1.7)], for instance.)

Prop Let $X = \operatorname{Spec} A$ be an affine scheme.
Then X noetherian scheme $\iff$ A noetherian ring

Proof "$\Leftarrow$" is clear by definition.
'$\Rightarrow$'. Let $X = \bigcup U_i$ an affine open cover with all U_i noetherian. Fix an index i. For $f \in A$ with $D(f) \subseteq U_i$, we have $D(f) = D_{U_i}(f|_{U_i}) = \operatorname{Spec}(\Gamma(U_i, \mathcal{O}_X)_{f|_{U_i}})$. Since localizations of noeth. rings are noeth., it follows that $D(f)$ is noetherian.
Replacing the U_i's by (several) $D(f_j)$'s we may therefore assume that each U_i is a principal open in X, say $U_i = D(f_i)$.

The following lemma then implies that every ideal $I \subseteq A$ is finitely generated, and hence that A is noetherian.

<u>Lemma</u> Let A be a ring, $f_1, \dots, f_r \in A$ s.t. $(f_1, \dots, f_r) = A$ $\left(\Leftrightarrow \bigcup_{i=1}^{r} D(f_i) = \operatorname{Spec} A\right)$.
Let M be an A-module such that for every $i = 1, \dots, r$ the localization M_{f_i} is a finitely generated A_{f_i}-module.
Then M is a finitely generated A-module.

<u>Proof</u> We write $M_{f_i} = \left\langle \frac{m_{ij}}{f_i^{n_j}}, j = 1, \dots, r_i \right\rangle_{A_{f_i}}$

and let $N := \langle m_{ij}, i, j \rangle \subseteq M$.

<u>Claim</u> $N = M$ (hence M fin. gen., as desired)

In fact, enough to show that $N_\mathfrak{p} = M_\mathfrak{p}$ for every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ (then $(M/N)_\mathfrak{p} = M_\mathfrak{p}/N_\mathfrak{p} = 0 \ \forall \mathfrak{p}$ which implies $M/N = 0$).

But for $\mathfrak{p} \in \operatorname{Spec} A$, say $\mathfrak{p} \in D(f_i)$, we have $N_\mathfrak{p} = (N_{f_i})_\mathfrak{p} = (M_{f_i})_\mathfrak{p} = M_\mathfrak{p}$.

Prop Let X be a (locally) noetherian scheme and let $U \subseteq X$ be an open subscheme. Then U is (locally) noetherian.

Proof • Clear for "locally noeth" since localizations of noetherian rings are noetherian, so principal open subschemes of Spec of a noeth. ring are noetherian.

• For "noetherian" use that every open subspace of a noetherian topological space is quasi-compact.

Cor X a locally noetherian scheme, $U \subseteq X$ affine open subscheme $\Rightarrow \Gamma(U, \mathcal{O}_X)$ is a noetherian ring.