# V. Projective space

# V.1 Gluing of schemes

Goal: describe how to "glue" a family $(U_i)_i$ of schemes, i.e. construct a scheme $X$ so that every $U_i$ "is" an open subscheme of $X$, and such that any two of these open subschemes intersect in a determined way.

**Def.** A glueing datum of schemes consists of the following data:
- a set $I$,
- a family $(U_i)_{i \in I}$ of schemes
- for all $i, j \in I$, an open subscheme $U_{ij} \subseteq U_i$ such that $U_{ii} = U_i$ for all $i$,
- for all $i, j \in I$, an isomorphism $\varphi_{ij} : U_{ji} \xrightarrow{\sim} U_{ij}$ such that for all $i, j, k \in I$,

"cocycle condition" $\quad \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} \quad$ on $U_{ki} \cap U_{kj}$

(in particular we require that $\varphi_{jk}(U_{ki} \cap U_{kj}) \subseteq U_{ji}$).

(it follows that for all $i$, $\varphi_{ii} = \mathrm{id}_{U_i}$,
for all $i,j$, $(\varphi_{ij})^{-1} = \varphi_{ji}$,
and $\varphi_{ij}$ induces an isomorphism
$U_{ji} \cap U_{jk} \cong U_{ij} \cap U_{ik}$).

<u>Prop</u> (Gluing of schemes) Given a gluing datum $((U_i)_{i\in I}, (U_{ij})_{i,j\in I}, (\varphi_{ij})_{i,j\in I})$, there exists a scheme $X$ together with open immersions $\psi_i : U_i \to X$ s.t. for all $i,j \in I$, $\psi_j \circ \varphi_{ji} = \psi_i$ on $U_{ij}$, $X = \bigcup_{i\in I} \psi_i(U_i)$, and

$$\psi_i(U_{ij}) = \psi_i(U_i) \cap \psi_j(U_j) = \psi_j(U_{ji}).$$

The scheme $X$ together with the $\psi_i$ is uniquely determined up to unique isomorphism.

21.12.2022

By "gluing of morphisms", $X$ together with the $\psi_i$ satisfies the following universal property:

For every scheme $T$ and every family $\xi_i : U_i \to T$ of morphisms such that $\xi_j \circ \varphi_{ji} = \xi_i$ for all $i,j$, there exists a unique morphism $\xi : X \to T$ with $\xi \circ \psi_i = \xi_i$ $\forall i \in I$.

In particular, this implies the uniqueness statement of the proposition.

<u>Proof</u> (of existence).

As a set, we define $X$ as the set $\coprod_{i \in I} U_i / \sim$ of equivalence classes for the following equivalence relation on the disjoint union $\coprod U_i$:

For $x_i \in U_i$, $x_j \in U_j$, $\quad x_i \sim x_j \iff x_i \in U_{ij}, x_j \in U_{ji}$, and $\varphi_{ji}(x_i) = x_j$.

The "cocycle condition" $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ implies that this is an equivalence relation.

Denote by $\psi_i : U_i \to \coprod U_i \twoheadrightarrow X$ the natural map (of sets). (Note that the $\psi_i$ are injective.)

We equip $X$ with the finest topology such that all the maps $\psi_i$ are continuous. More concretely, a subset $U \subseteq X$ is open if and only if for all $i \in I$, $\psi_i^{-1}(U) \subseteq U_i$ is open.

In particular, $X = \bigcup_{i \in I} \psi_i(U_i)$ is an open cover of $X$, and for all $i, j$, we have homeomorphisms

$$U_{ij} \xrightarrow{\sim} \psi_i(U_{ij}) = \psi_i(U_i) \cap \psi_j(U_j) = \psi_j(U_{ji}) \xleftarrow{\sim} U_{ji}.$$

To define a sheaf $\mathcal{O}_X$ of rings on $X$, recall that it is enough to define it on a basis of the topology of $X$. It is therefore enough to define $\mathcal{O}_X(U)$ (and restriction maps) for all $U \subseteq X$ open such that there exists $i \in I$ with $U \subseteq \psi_i(U_i)$. For each $U$ we fix such an index $i$ and define $\mathcal{O}_X(U) := \mathcal{O}_{U_i}(\psi_i^{-1}(U))$.

Whenever $U \subseteq \varphi_i(U_i) \cap \varphi_j(U_j)$ for $i, j \in I$, we can identify $\mathcal{O}_{U_i}(\varphi_i^{-1}(U))$ with $\mathcal{O}_{U_j}(\varphi_j^{-1}(U))$ using the isom. $\varphi_{ij}$, $\varphi_{ji}$. Making these identifications, $\mathcal{O}_X(U)$ is independent of the choices and in particular we obtain well-defined restriction maps.

Altogether we have defined a ringed space $(X, \mathcal{O}_X)$ s.t. for every $i$, $(U_i, \mathcal{O}_{U_i}) \cong (\varphi_i(U_i), \mathcal{O}_X|_{\varphi_i(U_i)})$.

In particular, $X$ is a scheme. By construction, it has all the properties stated in the proposition.

<u>Example</u> (Disjoint union) Given any family $(U_i)_{i \in I}$ of schemes, we can define a gluing datum by setting $U_{ij} := \emptyset$ for all $i, j$. The resulting scheme $X$ obtained by gluing is called the <u>disjoint union</u> of the schemes $U_i$ and denoted by $\coprod_{i \in I} U_i$.