I Introduction

11.10.22

Plan: Have a "long" introduction in order to

- provide some motivation for the (partly) more "technical" content that will come later

- give those participants who were not in the Algebra 2 class last term a little more time to brush up their commutative algebra knowledge
  - (prime) ideals, quotients
  - localization
  - spectrum of a ring, Zariski topology

further references on moodle page

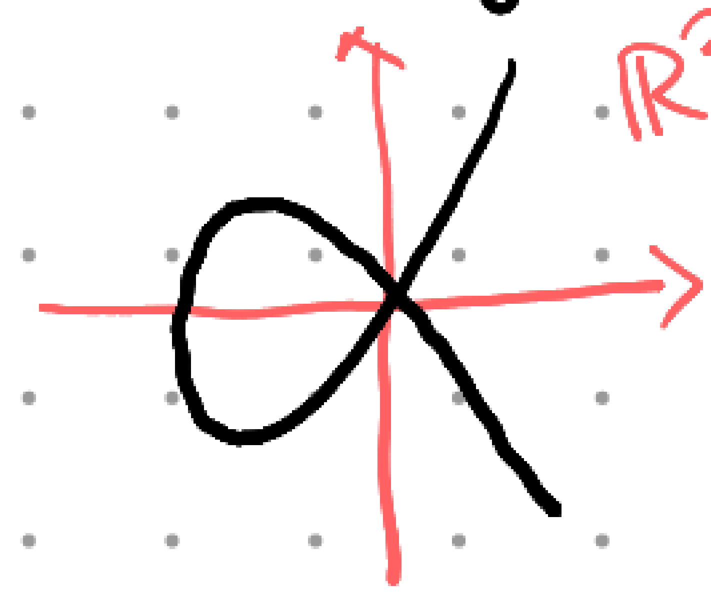
- What is algebraic geometry?
- (very) rough survey of this class
- I would like to know:
  What are your expectations? practical/organizational; regarding content

  also: what is your background/knowledge so far?

What is algebraic geometry?

→ study "geometric properties" of solution sets of systems of polynomial equations (over a field, or more generally a commutative ring)

Example $\{(x,y)\in\mathbb{R}^2;\ y^2 = x^2(x+1)\}$



Compared to previous/other courses:

linear algebra	algebra	algebraic geometry	algebraic number th.
systems of linear equations	one polynomial equation, one variable	several pol. equations, several variables	coefficients/solutions in $\mathbb{Z}$, $\mathbb{Q}$, $K/\mathbb{Q}$ finite, $\mathbb{F}_q$ ...

What does the "algebraic" in "algebraic geometry" refer to?

→ look at solutions/zero sets of polynomials (rather than, e.g., of (convergent) power series, differentiable/holomorphic functions)

→ use algebraic methods (commutative algebra)

"in principle" can work over arbitrary field

We start with a simple example which illustrates how geometric methods can be useful:

An (algebro-) geometric view on the theorem of Cayley-Hamilton

Theorem k a field, $A \in M_n(k)$. Then $\mathrm{charpol}_A(A) = 0$
($\in M_n(k)$).

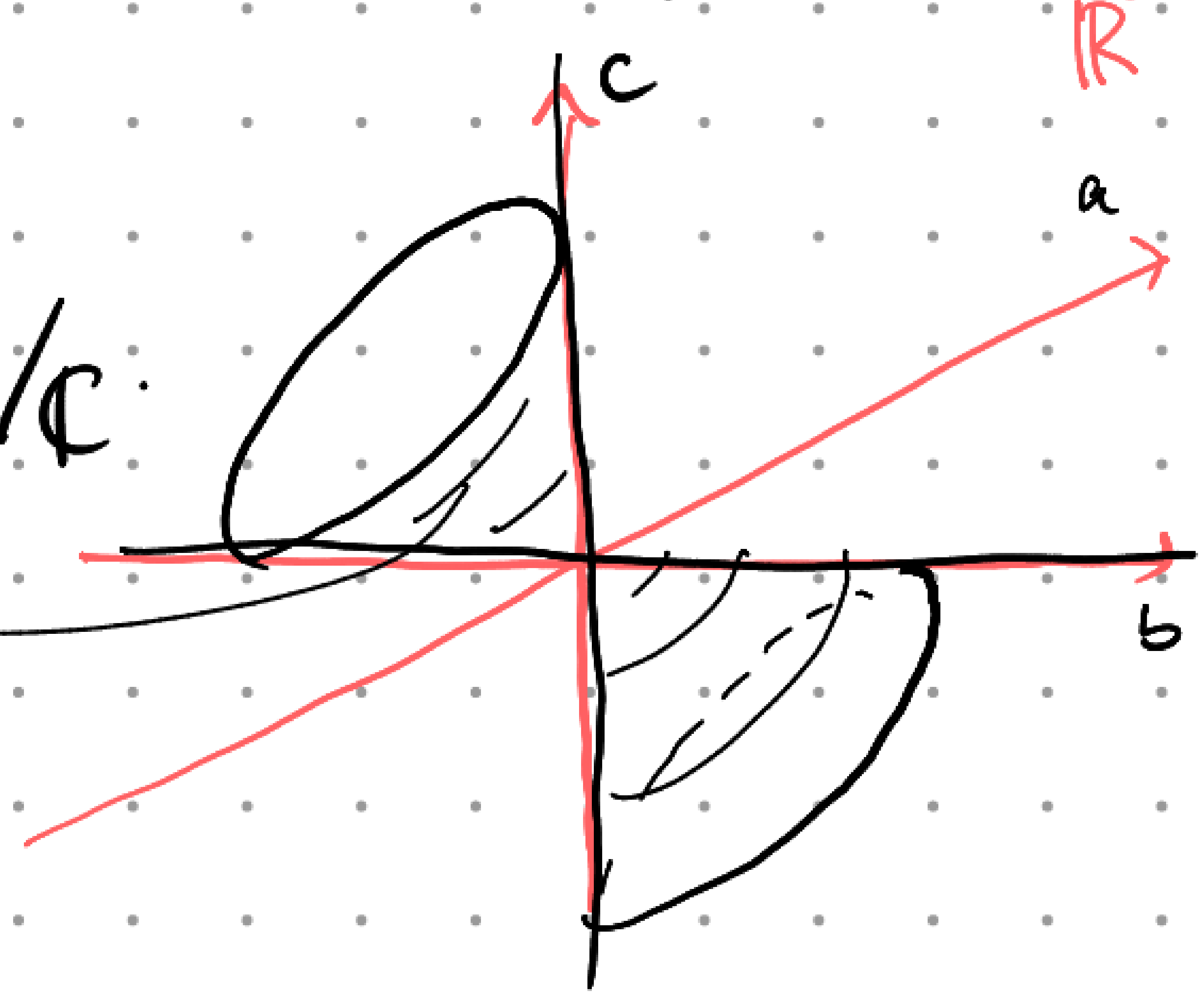
Let us consider the following situation: $k = \mathbb{R}$, [trace of A: $\mathrm{tr}(A) = 0$ — not really necessary, but simplifies the notation a little]

Want to use that the theorem is obviously true for diagonal matrices, and hence for diagonalizable matrices. In fact, suffices to have diag-able$/\mathbb{C}$.

So consider $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_2(\mathbb{R})^{tr=0} \underset{\text{as } \mathbb{R}\text{-v.s.}}{=} \mathbb{R}^3$

Have $\mathrm{charpol}_A = (T-a)(T+a) - bc = T^2 - (a^2 + bc)$

$\rightsquigarrow$ all matrices $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$
with $a^2 + bc \neq 0$ are diagonalizable$/\mathbb{C}$.



cone where $a^2 + bc = 0$

To prove the result for matrices $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2+bc=0$, consider the map

$$M_2(\mathbb{R})^{tr=0} \xrightarrow{\Phi} M_2(\mathbb{R}), \quad A \mapsto \mathrm{charpol}_A(A).$$

Goal: $\Phi(A) = 0$ for all A.

Can view Φ as a map $\mathbb{R}^3 \to \mathbb{R}^4$ given by polynomials, hence Φ is continuous.
Since $\{0\} \subset M_2(\mathbb{R})$ is closed, therefore $\Phi^{-1}(\{0\})$ is closed in $M_2(\mathbb{R})^{tr=0}$.

We have seen that $M_2(\mathbb{R})^{tr=0} \setminus V(a^2+bc) \subseteq \Phi^{-1}(\{0\})$.
Since $M_2(\mathbb{R})^{tr=0} \setminus V(a^2+bc)$ is dense in $M_2(\mathbb{R})^{tr=0}$, it follows that $\Phi^{-1}(\{0\}) = M_2(\mathbb{R})^{tr=0}$, as desired.

Question: How to deal with other fields?
(See Problem sheets 1, 2.)

The Zariski topology on k^n (k a field)

Since we want to study solution sets of systems of polynomial equations, let us introduce some notation:

k field, $f_1, \dots, f_m \in k[T_1, \dots, T_n]$

$$\rightsquigarrow V(f_1, \dots, f_m) = \{ (t_i)_i \in k^n ;\ \forall j = 1, \dots, m : f_j(t_1, \dots, t_n) = 0 \}$$

Furthermore, if K/k is any field extension, we can also plug elements of K^n into the f_j and define

$$V(f_1, \dots, f_m)(K) = \{ (t_i)_i \in K^n ;\ \forall j : f_j(t_1, \dots, t_n) = 0 \}.$$

(V stands for <u>v</u>anishing set (or in German: Verschwindungsmenge).)