Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2018/19

Problem sheet 8

Due date: Dec. 4, 2018.

Problem 33

Let X be a scheme and let $f \in \Gamma(X, \mathscr{O}_X)$. Show that

$$X_f := \{x \in X; f(x) \neq 0 \in \kappa(x)\}$$

is an open subset of X. Show that the image of f in $\Gamma(X_f, \mathscr{O}_X)$ is a unit in this ring.

Problem 34

Let p be a prime number. We say that a ring A *has characteristic* p if $p \cdot 1 = 0$ in A. Let $\iota\colon \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ be the canonical morphism. Let X be a scheme. Prove that the following are equivalent:

(1) The ring $\Gamma(X, \mathscr{O}_X)$ has characteristic p.

(2) For all open subsets $U \subseteq X$, the ring $\Gamma(U, \mathscr{O}_X)$ has characteristic p.

(3) The unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ factors as

$$X \longrightarrow \operatorname{Spec} \mathbb{F}_p \xrightarrow{\iota} \operatorname{Spec} \mathbb{Z}$$

If the conditions are satisfied, we say that X *has characteristic* p. Show that in this case the morphism $X \to \operatorname{Spec} \mathbb{F}_p$ is unique.

Give an example of a scheme X such that all residue class fields of the local rings of X have characteristic p, but such that X does not satisfy the above conditions.

Problem 35

Let X be a scheme of characteristic p. Show that there exists a unique morphism $(F, F^\flat)\colon X \to X$ of schemes such that on topological spaces, $F = \mathrm{id}_X$, and for an open $U \subseteq X$, $F^\flat$ is given by $\Gamma(U, \mathscr{O}_X) \to \Gamma(U, \mathscr{O}_X)$, $s \mapsto s^p$. This morphism is called the *absolute Frobenius morphism* of X.

Give an example where F is not an isomorphism.

Problem 36

Let k be an algebraically closed field, $Z = V(T_1, \dots, T_n) \subset \mathbb{A}^n_k$. Determine for which $n \geqslant 1$ the open subscheme $X := \mathbb{A}^n \smallsetminus Z$ is affine.