Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2018/19

Problem sheet 12

Due date: Jan. 15, 2018.

Problem 49

(1) Show that the properties "open immersion" and "closed immersion" are *local on the target*: Given a morphism $f\colon X \to Y$ and a cover $Y = \bigcup_i V_i$ by open subschemes, f is an open (closed) immersion if and only if for every i, the induced morphism $f^{-1}(V_i) \to V_i$ is an open (closed) immersion.

(2) Show that the properties "open immersion" and "closed immersion" are stable under composition of morphisms.

Problem 50

Show that open and closed immersions are *monomorphisms* in the category of schemes: If $f\colon X \to Y$ is an open (closed) immersion, then for every scheme S the induced map $\mathrm{Hom}(S, X) \to \mathrm{Hom}(S, Y)$ is injective.

Problem 51

(1) Let X be a scheme. Prove that there exists a unique reduced closed subscheme X_{red} of X which has the same underlying topological space as X.

(2) Let $f\colon X \to Y$ be a morphism of schemes. Show that f induces a unique morphism $f_{\mathrm{red}}\colon X_{\mathrm{red}} \to Y_{\mathrm{red}}$ such that the diagram

$$\begin{array}{ccc} X_{\mathrm{red}} & \hookrightarrow & X \\ {\scriptstyle f_{\mathrm{red}}}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{\mathrm{red}} & \hookrightarrow & Y \end{array}$$

Problem 52

Let k be a field, and let $A = k[X,Y]/(XY, X^2)$. Define two morphisms $f, g\colon \operatorname{Spec} A \to \operatorname{Spec} k[T]/(T^2)$ such that $f \neq g$, but such that there exists a non-empty open subset $U \subset \operatorname{Spec} A$ such that $f_{|U} = g_{|U}$.