Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2018/19

Problem sheet 1

Due date: Oct. 16, 2018.

Problem 1

A non-empty topological space X is called *irreducible*, if it is not equal to the union of two proper closed subsets.

1. Determine all topological spaces which are Hausdorff and irreducible. (Recall that a topological space X is called *Hausdorff*, if for any two points $u, v \in X$, $u \neq v$, there exist disjoint open subsets $U, V \subseteq X$ with $u \in U$, $v \in V$.

2. Let X be a non-empty topological space. Prove that the following properties are equivalent:

   (i) The space X is irreducible.

   (ii) Every non-empty open subset $U \subseteq X$ is dense in X (i.e., the smallest closed subset of X containing U is X).

   (iii) Every open subset $U \subseteq X$ is connected. (A topological space is called *connected*, if it cannot be written as the union of two disjoint proper closed subsets.)

   (iv) Any two non-empty open subsets of X have non-empty intersection.

Problem 2

Let k be an algebraically closed field.

(a) Let $n \geqslant 1$, $f \in k[T_1, \dots, T_n]$ a polynomial such that $f(t_1, \dots, t_n) = 0$ for all $t_1, \dots, t_n \in k$. Prove that $f = 0$. *Hint: You can use induction on n.*

(b) Prove that k^n (with the Zariski topology) is irreducible.

Problem 3

Let k be an algebraically closed field, and let $d \geqslant 1$. We identify the set of all monic polynomials $f(X) = X^d + t_{d-1}X^{d-1} + \cdots + t_0$ of degree d with k^d by mapping f to $(t_0, \ldots, t_{d-1})$.

Let $d = 2$. Prove that the subset of k^d corresponding to those polynomials which have a multiple zero is of the form $V(D)$ for a polynomial $D \in k[T_0, \ldots, T_{d-1}]$.

Remark. The same result holds for $d > 2$, but is more difficult to prove. One way to do it is roughly as follows: View f as a polynomial with coefficients in the field $K = k(t_0, \ldots, t_{d-1})$ of rational functions in d variables over k. Let L be the splitting field of f, a Galois extension of K. Let α_i be the zeros of f in L, and let $D = \prod_{i<j}(\alpha_i - \alpha_j)^2$. Then use the main theorem on elementary polynomials. Alternatively, use Galois theory to show that $D \in K$, and use that $k[t_0, \ldots, t_{d-1}]$ is integrally closed to conclude that $D \in k[t_0, \ldots, t_{d-1}]$.

Problem 4

Let k be an algebraically closed field. Let $n \geqslant 1$. We identify the space $M := \mathrm{Mat}_{n\times n}(k)$ of $(n \times n)$-matrices with entries in k with k^{n^2} and equip it with the Zariski topology. By Problem 2 (b), it is irreducible.

(a) Prove that the subset of M consisting of matrices A such that $\mathrm{charpol}_A(A) = 0$ is closed in M (without using the Theorem of Cayley-Hamilton).

(b) Use Problem 3 to prove that the subset of diagonalizable matrices with n different eigenvalues in k is open in M.

(c) Prove the Theorem of Cayley-Hamilton, i.e., prove that the subset in (a) equals all of M.