ALGEBRAIC GEOMETRY 2, SS 2019. LECTURE COURSE NOTES.

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Introduction

This lecture course is a continuation of the course *Algebraic Geometry 1* which covered the definition of schemes, and some basic notions about schemes and scheme morphisms: Reduced and integral schemes, immersions and subschemes, the functorial point of view, fiber products of schemes, separated and proper morphisms.

The main object of study of this term's course will be the notion of $\mathscr{O}_X$-module, a natural analogue of the notion of module over a ring in the context of sheaves of rings. As we will see, the $\mathscr{O}_X$-modules on a scheme X contain a lot of information about the geometry of this scheme, and we will study them using a variety of methods. In the second part, we will introduce the notion of *cohomology groups*, a powerful algebraic tool that makes its appearance in many areas of algebra and geometry.

These notes are not complete lecture notes (most proofs are omitted in the notes), but should rather be thought of as a rough summary of the content of the course.

1. $\mathscr{O}_X$-modules

General references: [GW] Ch. 7, [H] II.5. April 8, 2019

Definition and basic properties.

(1.1) Definition of $\mathscr{O}_X$-modules.

Definition 1.1. *Let $(X, \mathscr{O}_X)$ be a ringed space. An $\mathscr{O}_X$-module is a sheaf $\mathscr{F}$ of abelian groups on X together with maps*

$$\mathscr{O}_X(U) \times \mathscr{F}(U) \to \mathscr{F}(U) \qquad \textit{for each open} U \subseteq X$$

giving each $\mathscr{F}(U)$ the structure of an $\mathscr{O}_X(U)$-module, and which are compatible with the restriction maps for open subsets $U' \subseteq U \subseteq X$.

An $\mathscr{O}_X$-module homomorphism $\mathscr{F} \to \mathscr{G}$ between $\mathscr{O}_X$-modules $\mathscr{F}$, $\mathscr{G}$ on X is a sheaf morphism $\mathscr{F} \to \mathscr{G}$ such that for all open subsets $U \subseteq X$, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is a homomorphism of $\mathscr{O}_X(U)$-modules. We denote the set of $\mathscr{O}_X$-module homomorphisms from $\mathscr{F}$ to $\mathscr{G}$ by $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$; this is an $\mathscr{O}_X(X)$-module (and in particular an abelian group).

We obtain the category ($\mathscr{O}_X$-Mod) of $\mathscr{O}_X$-modules.

Remark 1.2. If $\mathscr{F}$ is an $\mathscr{O}_X$-module and $x \in X$, then the stalk $\mathscr{F}_x$ carries a natural $\mathscr{O}_{X,x}$-module structure. The $\kappa(x)$-vector space $\mathscr{F}(x) := \mathscr{F}_x \otimes_{\mathscr{O}_{X_x}} \kappa(x)$ is called the *fiber of $\mathscr{F}$ over x*.

Constructions, examples 1.3. Let X be a ringed space, $\mathscr{F}$ an $\mathscr{O}_X$-module.

(1) $\mathscr{O}_X$,
(2) submodules and quotients,
(3) $\oplus$, $\prod$, $\otimes$, (filtered) inductive limits,
(4) kernels, cokernels, image, exactness; these are compatible with passing to the stalks,
(5) restriction to open subsets: $\mathscr{F}_{X|U}$, $U \subseteq X$ open,
(6) $\mathscr{H}om$, $-^\vee$,

The category of $\mathscr{O}_X$-modules is an abelian category.

Definition 1.4. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module on the ringed space X. We call $\mathscr{F}$*

(a) free*, if it is isomorphic to $\bigoplus_{i\in I} \mathscr{O}_X$ for some set I,*
(b) locally free*, if there exists an open covering $X = \bigcup_j U_j$ of X such that $\mathscr{F}_{|U_j}$ is a free $\mathscr{O}_{U_j}$-module for each j.*

The rank *of a free $\mathscr{O}_X$-module is the cardinality of I as above (we usually regard it in $\mathbb{Z}\cup\{\infty\}$, without making a distinction between infinite cardinals). The* rank *of a locally free $\mathscr{O}_X$-module is a function $X \to \mathbb{Z} \cup \{\infty\}$ which is locally constant on X (i.e., on each connected component of X, there is an integer giving the rank).*

An invertible sheaf *or* line bundle *on X is a locally free sheaf of rank* 1.

For $\mathscr{L}$ invertible, there is a natural isomorphism $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}^\vee \cong \mathscr{O}_X$ (whence the name), cf. Problem 1. Hence $\otimes$ induces a group structure on the set of isomorphism classes of invertible sheaves in X. The resulting group is called the Picard group of X and denoted by $\mathrm{Pic}(X)$.

(1.2) Inverse image.

Definition 1.5. *Let $f\colon X \to Y$ be a morphism of ringed spaces, and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then $f_*\mathscr{F}$ carries a natural $\mathscr{O}_Y$-module structure and is called the* direct image *or* push-forward *of $\mathscr{F}$ under f.*

Definition 1.6. *Let $f\colon X \to Y$ be a morphism of ringed spaces, $\mathscr{F}$ an $\mathscr{O}_Y$-module.*

We define

$$f^*\mathscr{F} := f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

For $x \in X$, we have $(f^*\mathscr{F})_x \cong \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x}$.

We obtain functors f_*, f^* between the categories of $\mathscr{O}_X$-modules and $\mathscr{O}_Y$-modules.

Proposition 1.7. *Let $f\colon X \to Y$ be a morphism of ringed spaces. The functors f_* is right adjoint to the functor f^*:*

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F})$$

for all $\mathscr{O}_X$-modules $\mathscr{F}$, all $\mathscr{O}_Y$-modules $\mathscr{G}$, functorially in $\mathscr{F}$ and $\mathscr{G}$.

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Quasi-coherent $\mathscr{O}_X$-modules.

(1.3) The $\mathscr{O}_{\mathrm{Spec}\,A}$-module attached to an A-module M.

Definition 1.8. *Let A be a ring and M an A-module. Then setting*

$$D(f) \mapsto M_f, \quad f \in A,$$

is well-defined and defines a sheaf on the basis of principal open sets in $\mathrm{Spec}\,A$. *We denote the corresponding sheaf on* $\mathrm{Spec}\,A$ *by $\widetilde{M}$. It is an $\mathscr{O}_{\mathrm{Spec}\,A}$-module (by viewing each M_f as an A_f-module in the natural way).*

Remark 1.9. For an affine scheme X, in general not every $\mathscr{O}_X$-module has the above form. We will investigate this more closely soon.

Proposition 1.10. *Let A be a ring, and let M, N be A-modules. Then the maps*

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{\mathscr{O}_{\mathrm{Spec}\,A}}(\widetilde{M}, \widetilde{N})$$

given by

$$\varphi \mapsto \widetilde{f} := (\varphi_f\colon M_f \to N_f)_f$$

and, in the other direction,

$$\Phi \mapsto \Gamma(\mathrm{Spec}\,A, \Phi),$$

are inverse to each other. In other words, $\widetilde{\cdot}$ is a fully faithful functor from the category of A-modules to the category of $\mathscr{O}_{\mathrm{Spec}\,A}$-modules.

By applying the proposition to $M = A$, we also see that for an A-module N, $\widetilde{N}$ is zero if and only if N is zero.

The construction $M \mapsto \widetilde{M}$ is compatible with exactness, kernels, cokernels, images, direct sums, filtered inductive limits. (Cf. [GW] Prop. 7.14 for a more precise statement.)

(1.4) Quasi-coherent modules.

Definition 1.11. *Let X be a ringed space. An $\mathscr{O}_X$-module $\mathscr{F}$ is called* quasi-coherent*, if every $x \in X$ has an open neighborhood U such that there exists an exact sequence*

$$\mathscr{O}_U^{(J)} \to \mathscr{O}_U^{(I)} \to \mathscr{F}_{|U} \to 0$$

for suitable (possibly infinite) index sets I, J.

For a morphism $f\colon X \to Y$ of ringed spaces and a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$, the pull-back $f^*\mathscr{G}$ is a quasi-coherent $\mathscr{O}_X$-module (since f^{-1} is exact and tensor product is a right exact functor). The direct image f_* preserves the property of quasi-coherence (only) under certain conditions.

Locally free $\mathscr{O}_X$-modules are quasi-coherent.

Clearly, for a ring A and an A-module M, $\widetilde{M}$ is a quasi-coherent $\mathscr{O}_{\operatorname{Spec} A}$-module. We will see below that the converse is true as well:

For a ringed space X and $f \in \Gamma(X, \mathscr{O}_X)$, we write $X_f := \{x \in X;\ f_x \in \mathscr{O}_{X,x}^\times\}$, an open subset of X. We obtain a homomorphism

$$\Gamma(X, \mathscr{F})_f \to \Gamma(X_f, \mathscr{F})$$

for every $\mathscr{O}_X$-module $\mathscr{F}$.

Theorem 1.12. *Let X be a scheme and $\mathscr{F}$ an $\mathscr{O}_X$-module. The following are equivalent:*

(i) *For every affine open $\operatorname{Spec} A = U \subseteq X$, there exists an A-module M such that $\mathscr{F}_{|U} \cong \widetilde{M}$.*
(ii) *There exists a covering $X = \bigcup_i U_i$ by affine open subschemes $U_i = \operatorname{Spec} A_i$ and A_i-modules M_i such that $\mathscr{F}_{|U_i} \cong \widetilde{M_i}$ for all i.*
(iii) *The $\mathscr{O}_X$-module $\mathscr{F}$ is quasi-coherent.*
(iv) *For every affine open $\operatorname{Spec} A = U \subseteq X$ and every $f \in A$, the homomorphism $\Gamma(U, \mathscr{F})_f \to \Gamma(D(f), \mathscr{F})$ is an isomorphism.*

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Corollary 1.13. *Let A be a ring, $X = \operatorname{Spec} A$. The functor $\tilde{\cdot}$ induces an exact equivalence between the categories of A-modules and of quasi-coherent $\mathscr{O}_X$-modules.*

Corollary 1.14. *Let X be a scheme.*

(1) *Kernels, cokernels, images of $\mathscr{O}_X$-module homomorphisms between quasi-coherent $\mathscr{O}_X$-modules are quasi-coherent.*
(2) *Direct sums of quasi-coherent $\mathscr{O}_X$-modules are quasi-coherent.*

(3) *Let $\mathscr{F}$, $\mathscr{G}$ be quasi-coherent $\mathscr{O}_X$-module. Then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is quasi-coherent, and for every affine open $U \subseteq X$ we have*

$$\Gamma(U, \mathscr{F} \otimes \mathscr{G}) = \Gamma(U, \mathscr{F}) \otimes \Gamma(U, \mathscr{G}).$$

In particular, by (1) and (2) the category of quasi-coherent $\mathscr{O}_X$-module is an abelian category, and the inclusion functor into the category of all $\mathscr{O}_X$-modules preserves kernels and cokernels and direct sums.

(1.5) Direct and inverse image of quasi-coherent $\mathscr{O}_X$-module.

Proposition 1.15. *Let $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ be affine schemes, and let $f\colon X \to Y$ be a scheme morphism.*

(1) *Let N be an B-module, then $f_*(\widetilde{N}) = \widetilde{N_{[A]}}$ where $N_{[A]}$ is N, considered as an A-module via $\Gamma(f)\colon A \to B$.*

(2) *Let M be an A-module, then $f^*(\widetilde{M}) = \widetilde{M \otimes_A B}$.*

(1.6) Finiteness conditions.

Definition 1.16. *We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is* of finite type *(or* of finite presentation*, resp.), if every $x \in X$ has an open neighborhood $U \subseteq X$ such that there exists $n \geq 0$ (or $m, n \geq 0$, resp.) and a short exact sequence*

$$\mathscr{O}_X^n \to \mathscr{F} \to 0$$

(or

$$\mathscr{O}_X^m \to \mathscr{O}_X^n \to \mathscr{F} \to 0,$$

resp.).

On an affine scheme, this coincides with the corresponding definitions in terms of modules (via $M \mapsto \widetilde{M}$). Note that every $\mathscr{O}_X$-module of finite presentation is quasi-coherent. On a noetherian scheme, every quasi-coherent $\mathscr{O}_X$-module of finite type is of finite presentation.

Proposition 1.17. *Let X be a ringed space and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite presentation.*

(1) *For all $x \in X$ and for each $\mathscr{O}_X$-module $\mathscr{G}$, the canonical homomorphism of $\mathscr{O}_{X,x}$-modules*

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \to \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

is bijective.

(2) *Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules of finite presentation. Let $x \in X$ be a point and let $\theta\colon \mathscr{F}_x \xrightarrow{\sim} \mathscr{G}_x$ be an isomorphism of $\mathscr{O}_{X,x}$-modules. Then there exists an open neighborhood U of x and an isomorphism $u\colon \mathscr{F}_{|U} \xrightarrow{\sim} \mathscr{G}_{|U}$ of $\mathscr{O}_U$-modules with $u_x = \theta$.*

Proof. Problem 2. □

Proposition 1.18. *Let X be a ringed space, and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type. Then the support*

$$\operatorname{Supp}(\mathscr{F}) = \{x \in X;\ \mathscr{F}_x \neq 0\}$$

of $\mathscr{F}$ is a closed subset of X.

Proof. Problem 6. □

(1.7) Closed subschemes and quasi-coherent ideal sheaves.

Proposition 1.19. *Let X be a scheme. An ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_X$ defines a closed subscheme if and only if $\mathscr{I}$ is a quasi-coherent $\mathscr{O}_X$-module.*

We hence obtain an inclusion-reversing bijection between the set of closed subschemes of a scheme X and the set of quasi-coherent ideal sheaves in $\mathscr{O}_X$, mapping

- a quasi-coherent ideal sheaf $\mathscr{I}$ to $Z := (\operatorname{Supp}(\mathscr{O}_X/\mathscr{I}), i^{-1}(\mathscr{O}_X/\mathscr{I}))$, where $i\colon \operatorname{Supp}(\mathscr{O}_X/\mathscr{I}) \to X$ denotes the inclusion,
- a closed subscheme $Z \subseteq X$ to $\operatorname{Ker}(\mathscr{O}_X \to i_*\mathscr{O}_Z)$, where $i\colon Z \to X$ denotes the inclusion morphism.

We denote the closed subscheme corresponding to a quasi-coherent ideal sheaf $\mathscr{I}$ by $V(\mathscr{I})$.

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(1.8) Locally free sheaves on affine schemes.

There is an obvious "commutative algebra way" of writing down, for an A-module M, the condition that $\widetilde{M}$ is locally free.

Theorem 1.20. *Let A be a ring and M an A-module. Consider the following properties of M:*

(i) *$\widetilde{M}$ is a locally free $\mathscr{O}_{\operatorname{Spec} A}$-module.*
(ii) *M is locally free, i.e., there exist $f_1, \dots, f_n \in A$ generating the unit ideal such that for all i, the A_{f_i}-module M_{f_i} is free.*
(iii) *For all $\mathfrak{p} \in \operatorname{Spec} A$, the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is free.*
(iv) *The A-module M is flat.*
(1) *We have the implications* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).
(2) *If M is an A-module of finite presentation, then all the three properties are equivalent.*

Proof. Part (1) is easy. Part (2) is more difficult. The implication (iv) $\Rightarrow$ (iii) follows from Prop. 1.17. See [GW] Prop. 7.40. □

There is an obvious analogous theorem for $\mathscr{O}_X$-module on a scheme X, where we define

Definition 1.21. *Let X be a scheme. An $\mathscr{O}_X$-module $\mathscr{F}$ is called* flat*, if for all $x \in X$ the stalk $\mathscr{F}_x$ is a flat $\mathscr{O}_{X,x}$-module.*

More generally, given an $\mathscr{O}_X$-module $\mathscr{F}$ and a morphism $f\colon X \to Y$ we say that $\mathscr{F}$ is f-flat or flat over Y, if for all $x \in X$ the stalk $\mathscr{F}_x$ is a flat $\mathscr{O}_{Y,f(x)}$-module (via $f_x^\sharp\colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$).

If A is a domain, then every flat A-module M is torsion-free (i.e., multiplication by s is injective for all $s \in A \setminus \{0\}$). The converse holds only rarely; it does hold if A is a principal ideal domain and M is finitely generated.

Remark 1.22.

(1) Let A be a principal ideal domain. Then every finitely generated locally free (in the sense of condition (i′) in the theorem) A-module is free. (Use the structure theorem for finitely generated modules over principal ideal domains.)

(2) It is a difficult theorem (conjectured by Serre, proved independently by Quillen and Suslin) that every locally free sheaf of finite type on $\mathbb{A}^n_k$, k a field, is free. The same statement holds even for k a discrete valuation ring.

(3) It will not be relevant in the course, but to complete the picture (and since I did not remember the correct statement during the lecture ...), let us remark that one can show that in the previous two items the hypothesis *of finite type* can be omitted. In fact, whenever R is a ring which is noetherian and such that $\operatorname{Spec} R$ is connected, then every locally free R-module which is *not finitely generated* is free. One way to show this to combine the paper [Ba] by H. Bass with the difficult theorem that the property of a module of being "projective" can be checked Zariski-locally on $\operatorname{Spec} A$ ([Stacks] `058B`), which shows that all locally free R-modules, finitely generated or not, are projective. Maybe there is also a more direct way, without talking about projective modules?

(4) Let A be a noetherian unique factorization domain. Then every invertible sheaf on $\operatorname{Spec} A$ is free.

(5) See the answers to this question (`mathoverflow.net/q/54356`) for examples of non-free locally free modules over $\operatorname{Spec} A$ for factorial (and even, in addition, regular) noetherian rings A.

(6) Let A be a domain, and let M be a locally free A-module of rank 1. Then M is isomorphic to a *fractional ideal*, i.e., to a finitely generated sub-A-module of $K := \operatorname{Frac}(A)$. (Cf. Problem 8 for a converse statement in the case that A is a Dedekind domain.)

2. Line bundles and divisors

General references: [GW] Ch. 11, in particular (11.9), (11.13); [H] II.6.

A *divisor* on a scheme X should be thought of an object that encodes a "configuration of zeros and poles (with multiplicities)" that a function on X

could have. Below, we will see two ways to make this precise and compare them.

Let X be an integral (i.e., reduced and irreducible) scheme. We denote by $K(X)$ the field of rational functions of X.

Later we will impose the additional condition that X is noetherian and that all local rings $\mathscr{O}_{X,x}$ are unique factorization domains.

An important example that is good to keep in mind is the case of a *Dedekind scheme of dimension* 1, i.e., X is a noetherian integral scheme such that all points except for the generic point are closed, and such that for every closed point $x \in X$ the local ring $\mathscr{O}_{X,x}$ is a principal ideal domain (in other words: all local rings are discrete valuation rings), and the generic point is not closed itself. If a Dedekind scheme X is a k-scheme of finite type for some algebraically closed (or at least perfect) field k, then we call X a smooth algebraic curve over k.

Cartier divisors.

(2.1) Cartier divisors: Definition.

Denote by $K(X) = \mathscr{O}_{X,\eta}$ the field of rational functions on the integral scheme X, where $\eta \in X$ is the generic point. We denote by $\mathscr{K}_X$ the constant sheaf with value $K(X)$, i.e., $\mathscr{K}_X(U) = K(X)$ for all $\emptyset \neq U \subseteq X$ open. Since X is irreducible, this is a sheaf.

The notion of Cartier divisor encodes a zero/pole configuration by specifying, *locally on* X, functions with the desired zeros and poles. Since functions which are units in $\Gamma(U, \mathscr{O}_X)$ should be regarded as having no zeros and/or poles on U, we consider functions only up to units.

Definition 2.1. *A* Cartier divisor *on X is given by a tuple $(U_i, f_i)_i$, where $X = \bigcup_i U_i$ is an open cover, $f_i \in K(X)^\times$, and $f_i/f_j \in \Gamma(U_i \cap U_J, \mathscr{O}_X)^\times$ for all i, j. Two such tuples $(U_i, f_i)_i$, $(Vj, g_j)_j$ give rise to the same divisor, if $f_i g_j^{-1} \in \Gamma(U_i \cap V_j, \mathscr{O}_X)^\times$ for all i, j.*

With addition given by

$$(U_i, f_i)_i + (V_j, g_j)_j = (U_i \cap V_j, f_i g_j)_{i,j}$$

the set $\mathrm{Div}(X)$ of all Cartier divisors on X is an abelian group.

Definition 2.2. *A Cartier divisor of the form (X, f), $f \in K(X)^\times$, is called a* principal *divisor. Divisors D, D' on X are called* linearly equivalent*, if $D - D'$ is a principal divisor. The set of principal divisors is a subgroup of* $\mathrm{Div}(X)$ *and the quotient* $\mathrm{DivCl}(X)$ *of* $\mathrm{Div}(X)$ *by this subgroup is called the* divisor class group oX.

(2.2) The line bundle attached to a Cartier divisor.

Let D be a Cartier divisor on X. We define an invertible $\mathscr{O}_X$-module $\mathscr{O}_X(D)$ as follows:

$$\Gamma(U, \mathscr{O}_X(D)) = \{f \in K(X); \forall i : f_i f \in \Gamma(U \cap U_i, \mathscr{O}_X)\} \quad \text{for } \emptyset \neq U \subseteq X \text{ open.}$$

For each i, we have $\mathscr{O}_X(D)_{|U_i} f_i^{-1} \mathscr{O}_{U_i} \subset \mathscr{K}_X$, so multiplication by f_i gives an $\mathscr{O}_{U_i}$-module isomorphism $\mathscr{O}_X(D)_{|U_i} \cong \mathscr{O}_{U_i}$.

Proposition 2.3. *The map $D \mapsto \mathscr{O}_D(X)$ induces group isomorphisms $\mathrm{Div}(X) \cong \{\mathscr{L} \subset \mathscr{K}_X$ invertible $\mathscr{O}_X$-module$\}$ and $\mathrm{DivCl}(X) \cong \mathrm{Pic}(X)$.*

To get a more geometric view on divisors, a first step is the following definition of the support of a divisor. We will carry this further by introducing the notion of Weil divisor, see below, and relating it to Cartier divisors.

Definition 2.4. *The* support *of a Cartier divisor D is*

$$\mathrm{Supp}(D) = \{x \in X;\ f_{i,x} \in K(X)^\times \setminus \mathscr{O}_{X,x}^\times \text{ (wherex} \in U_i)\},$$

a proper closed subset of X.

Weil divisors.

Now let X be a noetherian integral scheme such that all local rings $\mathscr{O}_{X,x}$ are factorial. April 24, 2019

(2.3) Definition of Weil divisors.

Let $Z^1(X)$ denote the free abelian group on maximal proper integral subschemes of X (equivalently: those integral subschemes $Z \subset X$ such that for the generic point $\eta_Z \in Z$ we have $\dim \mathscr{O}_{X,\eta_Z} = 1$). We say that Z *has codimension* 1. We also write $\mathscr{O}_{X,Z} := \mathscr{O}_{X,\eta_Z}$.

By our assumptions on X, all the rings $\mathscr{O}_{X,Z}$ are discrete valuation rings. (Since they are noetherian domains of dimension 1 by assumption, it is equivalent to require that they are integrally closed, or factorial, or that they are regular.) We denote by $v_Z \colon K(X)^\times \to \mathbb{Z}$ the corresponding discrete valuation on K, and set $v_Z(0) = \infty$.

Definition 2.5. *An element of $Z^1(X)$ is called a* Weil divisor. *We write Weil divisors as finite "formal sums" $\sum n_Z[Z]$ where $Z \subset X$ runs through the integral closed subschemes of X of codimension 1.*

For $f \in K(X)^\times$, we define the divisor attached to f as

$$\mathrm{div}(f) = \sum_Z v_Z(f)[Z].$$

Note that the sum is finite, i.e., $v_Z(f) = 0$ for all but finitely many Z. Weil divisors of this form are called *principal Weil divisors*. Two Weil divisors are called *linearly equivalent*, if their difference is a principal divisor.

(2.4) Weil divisors vs. Cartier divisors.

Generalizing the definition of principal divisors, we can construct a group homomorphism $\mathrm{cyc}\colon \mathrm{Div}(X) \to Z^1(X)$ as follows:

$$D = (U_i, f_i) \mapsto \sum v_Z(f_{i_Z})[Z],$$

where for each Z we choose an index i_Z so that U_{i_Z} contains the generic point of Z (equivalently: $U_{i_Z} \cap Z \neq \emptyset$).

Proposition 2.6. *The map* cyc *is a group isomorphism* $\mathrm{Div}(X) \cong Z^1(X)$. *Under this isomorphism, the subgroups of principal divisors on each side correspond to each other, whence it induces an isomorphism* $\mathrm{DivCl}(X) \cong \mathrm{Cl}(X) \cong \mathrm{Pic}(X)$.

(2.5) The theorem of Riemann and Roch.

No proofs were given in the lecture for the following results.
Reference: [H] IV.1.

Now let X a Dedekind scheme which is a scheme of finite type over an algebraically closed field k. In addition we assume that there exist $n \geq 1$ and a closed immersion $X \hookrightarrow \mathbb{P}^n_k$.

For a (Weil) divisor $D = \sum_Z n_Z[Z]$ we define the *degree* $\deg(D)$ of D as $\deg(D) := \sum_Z n_Z$. We obtain a group homomorphism $Z^1(X) \to \mathbb{Z}$. Under our assumption that X is a closed subscheme of some projective space, one can show that this homomorphism factors through $\mathrm{Cl}(X)$:

Theorem 2.7. *Let $f \in K(X)$. Then* $\deg(\mathrm{div}(f)) = 0$.

We can now state (a simplified version of) the Theorem of Riemann–Roch. For a divisor D we write $\ell(D) = \dim_k \Gamma(X, \mathscr{O}_X(D))$.

Proposition 2.8. *For each D, $\ell(D)$ is finite. If $\ell(D) \geq 0$, then* $\deg(D) \geq 0$.

Theorem 2.9. (Riemann-Roch) *For X as above, there exist $g \in \mathbb{Z}_{\geq 0}$ and $K \in \mathrm{Div}(X)$ such that for every divisor D on X, we have*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

Corollary 2.10. *In the above situation, we have*
(1) $\ell(K) = g$,
(2) $\deg(K) = 2g - 2$,
(3) *for every D with* $\deg(D) > 2g - 2$, *we have* $\ell(D) = \deg(D) + 1 - g$.

The number g is called the *genus* of the curve X. For X as above which is of the form $V_+(f) \subset \mathbb{P}^2_k$, there is the following formula for the genus:

Proposition 2.11. *Let X as above be of the form $V_+(f) \subset \mathbb{P}^2_k$ for a homogeneous polynomial f of degree d. Then the genus g of X is given by*

$$g = \frac{(d-1)(d-2)}{2}.$$

3. Smoothness and differentials

General reference: [GW] Ch. 6. April 29, 2019

The Zariski tangent space.

(3.1) Definition of the Zariski tangent space.

Definition 3.1. *Let X be a scheme, $x \in X$, $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ the maximal ideal in the local ring at x, $\kappa(x)$ the residue class field of X in x. The $\kappa(x)$-vector space $(\mathfrak{m}/\mathfrak{m}^2)^*$ is called the* (Zariski) tangent space of X in x.

Definition 3.2. *Let R be a ring, $f_1, \dots, f_r \in R[T_1, \dots, T_n]$. We call the matrix*

$$J_{f_1,\dots,f_r} := \left(\frac{\partial f_i}{\partial T_j}\right)_{i,j} \in M_{r\times n}(R[T_\bullet])$$

the Jacobian matrix *of the polynomials f_i. Here the partial derivatives are to be understood in a formal sense.*

Remark 3.3.

(1) If in the above setting the ideal $\mathfrak{m}$ is finitely generated, then $\dim_{\kappa(x)} T_x X$ is the minial number of elements needed to generate $\mathfrak{m}$ and in particular is finite.

(2) The tangent space construction if functorial in the following sense: Given a scheme morphism $f\colon X \to Y$ and $x \in X$ such that $\dim_{\kappa(x)} T_x X$ is finite or $[\kappa(x) : \kappa(f(x))]$ is finite, then we obtain a map

$$df_x\colon T_x X \to T_{f(x)} Y \otimes_{\kappa(f(x))} \kappa(x).$$

Example 3.4. Let k be a field, $X = V(f_1, \dots, f_m) \subseteq \mathbb{A}^n_k$, $f_i \in k[T_1, \dots, T_n]$, $x = (x_i)_i \in k^n = \mathbb{A}^n(k)$. Then there is a natural identification $T_x X = \operatorname{Ker}(J_{f_1,\dots,f_m}(x))$, where $J_{f_1,\dots,f_m}(x)$ denotes the matrix with entries in $\kappa(x) = k$ obtained by mapping each entry of $J_{f_1,\dots,f_m}$ to $\kappa(x)$, which amounts to evaluating these polynomials at x.

Proposition 3.5. *Let k be a field, X a k-scheme, $x \in X(k)$. There is an identification (functorial in (X, x))*

$$X(k[\varepsilon]/(\varepsilon^2))_x := \{f \in \operatorname{Hom}_k(\operatorname{Spec} k[\varepsilon]/(\varepsilon^2), X);\ \operatorname{im}(f) = \{x\}\} = T_x X.$$

Smooth morphisms.

(3.2) Definition of smooth morphisms.

Definition 3.6. *A morphism $f\colon X \to Y$ of schemes is called* smooth of relative dimension $d \geq 0$ in $x \in X$, *if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \operatorname{Spec} R \subseteq Y$ of $f(x)$ such that $f(U) \subseteq V$ and an open immersion $j\colon U \to \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d})$ such that the triangle*

$$\begin{array}{ccc} U & \xrightarrow{\ j\ } & \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d}) \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & V & \end{array}$$

is commutative, and that the Jacobian matrix $J_{f_1,\dots,f_{n-d}}(x)$ has rank $n-d$.

We say that $f\colon X \to Y$ is smooth of relative dimension d *if f is smooth of relative dimension d at every point of X. Instead of* smooth of relative dimension 0*, we also use the term* Ã©tale.

With notation as above, if f is smooth at $x \in X$, then x as an open neighborhood such that f is smooth at all points of this open neighborhood. Clearly, $\mathbb{A}^n_S$ and $\mathbb{P}^n_S$ are smooth of relative dimension n for every scheme S. (It is harder to give examples of non-smooth schemes directly from the definition; we will come back to this later.)

May 6, 2019

(3.3) Dimension of schemes.

Recall from commutative algebra that for a ring R we define the (Krull) dimension $\dim R$ of R as the supremum over all lengths of chains of prime ideals, or equivalently as the dimension of the topological space $\operatorname{Spec} R$ in the sense of the following definition.

Definition 3.7. *Let X be a topological space. We define the dimension of X as*

$$\dim X := \sup\{\ell;\ \textit{there exists a chain } Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_\ell \\ \textit{of closed irreducible subsets } Z_i \subseteq X\}.$$

We will use this notion of dimension for non-affine schemes, as well. Recall the following theorem about the dimension of finitely generated algebras over a field from commutative algebra:

Theorem 3.8. *Let k be a field, and let A be a finitely generated k-algebra which isi a domain. Let $\mathfrak{m} \subset A$ be a maximal ideal. Then*

$$\dim A = \operatorname{trdeg}_k(\operatorname{Frac}(A)) = \dim A_{\mathfrak{m}}.$$

By passing to an affine cover, we obtain the following corollary:

Corollary 3.9. *Let k be a field, and let X be an integral k-scheme which is of finite type over k. Denote by $K(X)$ its field of rational functions. Let $U \subseteq X$ be a non-empty open subset, and let $x \in X$ be a closed point. Then*

$$\dim X = \dim U = \mathrm{trdeg}_k(K(X)) = \dim \mathscr{O}_{X,x}.$$

(3.4) Existence of smooth points.

Let k be a field.

Lemma 3.10. *Let X, Y be [integral[1]] k-schemes which are locally of finite type over k. Let $x \in X$, $y \in Y$, and let $\varphi\colon \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ be an isomorphism of k-algebras. Then there exist open neighborhoods U of x and V of y and an isomorphism $h\colon U \to V$ of k-schemes with $h_x^\sharp = \varphi$.*

Proposition 3.11. *Let X be an integral k-scheme of finite type. Assume that $K(X) \cong k(T_1, \dots, T_d)[\alpha]$ with α separable algebraic over $k(T_1, \dots, T_d)$. (This is always possible if k is perfect.) (Then $\dim X = d$ by the above.)*

Then there exists a dense open subset $U \subseteq X$ and a separable irreducible polynomial $g \in k(T_1, \dots, T_d)[T]$ with coefficients in $k[T_1, \dots, T_d]$, such that U is isomorphic to a dense open subset of $\operatorname{Spec} k[T_1, \dots T_d]/(g)$.

Theorem 3.12. *Let k be a perfect field, and let X be a nonempty reduced k-scheme locally of finite type over k. Then the smooth locus*

$$X_{\mathrm{sm}} := \{x \in X;\ X \to \operatorname{Spec} k \text{ is smooth in} x\}$$

of X is open and dense.

May 8, 2019

(3.5) Regular rings.

For references to the literature, see [GW] App. B, in particular B.73, B.74, B.75

Definition 3.13. *A noetherian local ring A with maximal ideal $\mathfrak{m}$ and residue class field κ is called* regular*, if $\dim A = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.*

One can show that the inequality $\dim A \leq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ always holds. Therefore we can rephrase the definition as saying that A is regular if $\mathfrak{m}$ has a generating system consisting of $\dim A$ elements.

Definition 3.14. *A noetherian ring A is called* regular*, if $A_\mathfrak{m}$ is regular for every maximal ideal $\mathfrak{m} \subset A$.*

We quote the following (mostly non-trivial) results about regular rings:

[1]The statement is true in general, but in the lecture we proved it only with the additional assumption that X and Y are integral.

Theorem 3.15.

(1) *Every localization of a regular ring is regular.*
(2) *If A is regular, then the polynomial ring $A[T]$ is regular.*
(3) *(Theorem of Auslander–Buchsbaum) Every regular local ring is factorial.*
(4) *Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and of dimension d, and let $f_1, \dots, f_r \in \mathfrak{m}$. Then $A/(f_1, \dots, f_r)$ is regular of dimension $d-r$ if and only if the images of the f_i in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $A/\mathfrak{m}$.*

(3.6) Smoothness and regularity.

Let k be a field.

Lemma 3.16. *Let X be a k-scheme locally of finite type. Let $x \in X$ such that $X \to \operatorname{Spec} k$ is smooth of relative dimension d in x. Then $\mathscr{O}_{X,x}$ is regular of dimension $\leq d$. If moreover x is closed, then $\mathscr{O}_{X,x}$ is regular of dimension d.*

Lemma 3.17. *Let $X = V(g_1, \dots, g_s) \subseteq \mathbb{A}^n_k$, and let $x \in X$ be a closed point. If $\operatorname{rk} J_{g_1,\dots,g_s}(x) \geq n - \dim \mathscr{O}_{X,x}$, then x is smooth in X/k, and $\operatorname{rk} J_{g_1,\dots,g_s}(x) = n - \dim \mathscr{O}_{X,x}$.*

Theorem 3.18. *Let X be a k-scheme locally of finite type, $x \in X$ a closed point, $d \geq 0$. Fix an algebraically closed extension field K of k and write $X_K = X \otimes_k K$. The following are equivalent:*

(i) *The morphism $X \to \operatorname{Spec} k$ is smooth of relative dimension d at x.*
(ii) *For all points $\overline{x} \in X_K$ lying over x, X_K is smooth over K of relative dimension d at $\overline{x}$.*
(iii) *There exists a point $\overline{x} \in X_K$ lying over x, such that X_K is smooth over K of relative dimension d at $\overline{x}$.*
(iv) *For all points $\overline{x} \in X_K$ lying over x, the local ring $\mathscr{O}_{X_K,\overline{x}}$ is regular of dimension d.*
(v) *There exists a point $\overline{x} \in X_K$ lying over x, such that the local ring $\mathscr{O}_{X_K,\overline{x}}$ is regular of dimension d.*

If these conditions hold, then the local ring $\mathscr{O}_{X,x}$ is regular of dimension d, and if $\kappa(x) = k$, then this last condition is equivalent to the previous ones.

May 13, 2019

Corollary 3.19. *Let X be an irreducible scheme of finite type over k, and let $x \in X(k)$ be a k-valued point. Then $X \to \operatorname{Spec} k$ is smooth at x if and only if $\dim X = \dim_k T_x X$.*

Corollary 3.20. *Let $X = V(g_1, \dots, g_s) \subseteq \mathbb{A}^n_k$ and let $x \in X$ be a smooth closed point. Let $d = \dim \mathscr{O}_{X,x}$. Then $J_{g_1,\dots,g_s}(x)$ has rank $n - d$. In particular, $s \geq n - d$.*

After renumbering the g_i, if necessary, there exists an open neighborhood U of x and an open immersion $U \subseteq V(g_1, \dots, g_{n-d})$, i.e., locally around x, "X is cut out in affine space by the expected number of equations".

Corollary 3.21. *Let X be locally of finite type over k. The following are equivalent:*
(i) *X is smooth over k.*
(ii) *$X \otimes_k L$ is regular for every field extension L/k.*
(iii) *There exists an algebraically closed extension field K of k such that $X \otimes_k K$ is regular.*

The sheaf of differentials.

General references: [M2] §25, [Bo] Ch. 8, [H] II.8.

(3.7) Modules of differentials.

Let A be a ring.

Definition 3.22. *Let B be an A-algebra, and M a B-module. An A-*derivation *from B to M is a homomorphism $D\colon B \to M$ of abelian groups such that*
(a) *(Leibniz rule) $D(bb') = bD(b') + b'D(b)$ for all $b, b' \in B$,*
(b) *$d(a) = 0$ for all $a \in A$.*

Assuming property (a), property (b) is equivalent to saying that D is a homomorphism of A-modules. We denote the set of A-derivations $B \to M$ by $\mathrm{Der}_A(B, M)$; it is naturally a B-module.

Definition 3.23. *Let B be an A-algebra. We call a B-module $\Omega_{B/A}$ together with an A-derivation $d_{B/A}\colon B \to \Omega_{B/A}$ a module of (relative, KÃ¤hler) differentials of B over A if it satisfies the following universal property:*

For every B-module M and every A-derivation $D\colon B \to M$, there exists a unique B-module homomorphism $\psi\colon \Omega_{B/A} \to M$ such that $D = \psi \circ d_{B/A}$.

In other words, the map $\mathrm{Hom}_B(\Omega_{B/A}, M) \to \mathrm{Der}_A(B, M)$, $\psi \mapsto \psi \circ d_{B/A}$ is a bijection.

Lemma 3.24. *Let I be a set, $B = A[T_i, i \in I]$ the polynomial ring. Then $\Omega_{B/A} := B^{(I)}$ with $d_{B/A}(T_i) = e_i$, the "i-th standard basis vector" is a module of differentials of B/A.*

So we can write $\Omega_{B/A} = \bigoplus_{i \in I} B d_{B/A}(T_i)$.

Lemma 3.25. *Let $\varphi\colon B \to B'$ be a surjective homomorphism of A-algebras, and write $\mathfrak{b} = \mathrm{Ker}(\varphi)$. Assume that a module of differentials $(\Omega_{B/A}, d_{B/A})$*

for B/A exists. Then

$$\Omega_{B/A}/(\mathfrak{b}\Omega_{B/A} + B' d(\mathfrak{b}))$$

together with the derivation $d_{B'/A}$ induced by $d_{B/A}$ is a module of differentials for B'/A.

Corollary 3.26. *For every A-algebra B, a module $\Omega_{B/A}$ of differentials exists. It is unique up to unique isomorphism.*

May 15, 2019

We will see later that for a scheme morphism $X \to Y$, one can construct an $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with a "derivation" $\mathscr{O}_X \to \Omega_{X/Y}$ by gluing sheaves associated to modules of differentials attached to the coordinate rings of suitable affine open subschemes of X and Y.

Let $\varphi\colon A \to B$ be a ring homomorphism. For the next definition, we will consider the following situation: Let C be a ring, $I \subseteq C$ an ideal with $I^2 = 0$, and let

$$\begin{array}{ccc} B & \longrightarrow & C/I \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

be a commutative diagram (where the right vertical arrow is the canonical projection). We will consider the question whether for these data, there exists a homomorphism $B \to C$ (dashed in the following diagram) making the whole diagram commutative:

$$\begin{array}{ccc} B & \longrightarrow & C/I \\ {\scriptstyle\varphi}\uparrow & \searrow & \uparrow \\ A & \longrightarrow & C \end{array}$$

Definition 3.27. *Let $\varphi\colon A \to B$ be a ring homomorphism.*

(1) *We say that φ is* formally unramified*, if in every situation as above, there exists at most one homomorphism $B \to C$ making the diagram commutative.*
(2) *We say that φ is* formally smooth*, if in every situation as above, there exists at least one homomorphism $B \to C$ making the diagram commutative.*
(3) *We say that φ is* formally Ã©tale*, if in every situation as above, there exists a unique homomorphism $B \to C$ making the diagram commutative.*

Passing to the spectra of these rings, we can interpret the situation in geometric terms: $\operatorname{Spec} C/I$ is a closed subscheme of $\operatorname{Spec} C$ with the same topological space, so we can view the latter as an "infinitesimal thickening" of the former. The question becomes the question whether we can extend the morphism from $\operatorname{Spec} C/I$ to $\operatorname{Spec} B$ to a morphism from this thickening.

Proposition 3.28. *Let $\varphi\colon A\to B$ be a ring homomorphism. Then φ is formally unramified if and only if $\Omega_{B/A}=0$.*

For an algebraic field extension L/K one can show that $K\to L$ is formally unramified if and only if it is formally smooth if and only if L/K is separable. Cf. Problem 27 and [M2] §25, §26 (where the discussion is extended to the general, not necessarily algebraic, case).

Theorem 3.29. *Let $f\colon A\to B$, $g\colon B\to C$ be ring homomorphisms. Then we obtain a natural sequence of C-modules*

$$\Omega_{B/A}\otimes_B C\to\Omega_{C/A}\to\Omega_{C/B}\to 0$$

which is exact.

If moreover g is formally smooth, then the sequence

$$0\to\Omega_{B/A}\otimes_B C\to\Omega_{C/A}\to\Omega_{C/B}\to 0$$

is a split short exact sequence.

May 20, 2019

Theorem 3.30. *Let $f\colon A\to B$, $g\colon B\to C$ be ring homomorphisms. Assume that g is surjective with kernel $\mathfrak{b}$. Then we obtain a natural sequence of C-modules*

$$\mathfrak{b}/\mathfrak{b}^2\to\Omega_{B/A}\otimes_B C\to\Omega_{C/A}\to 0,$$

where the homomorphism $\mathfrak{b}/\mathfrak{b}^2\to\Omega_{B/A}\otimes_B C$ is given by $x\mapsto d_{B/A}(x)\otimes 1$.

If moreover $g\circ f$ is formally smooth, then the sequence

$$0\to\mathfrak{b}/\mathfrak{b}^2\to\Omega_{B/A}\otimes_B C\to\Omega_{C/A}\to 0$$

is a split short exact sequence.

(3.8) The sheaf of differentials of a scheme morphism.

Remark 3.31. Let again B an A-algebra. There is the following alternative construction of $\Omega_{B/A}$: Let $m\colon B\otimes_A B\to B$ be the multiplication map, and let $I=\mathrm{Ker}(m)$. Then I/I^2 is a B-module, and $d\colon B\to I/I^2$, $b\mapsto 1\otimes b-b\otimes 1$, is an A-derivation. One shows that $(I/I^2,d)$ satisfies the universal property defining $(\Omega_{B/A},d_{B/A})$.

We can use a similar definition as we used for ring homomorphisms above to define the notions of formally unramified, formally smooth and formally Ã©tale morphisms of schemes.

Definition 3.32. *Let $f\colon X\to Y$ be a morphism of schemes.*

(1) *We say that f is* formally unramified, *if for every ring C, every ideal I with $I^2=0$, and every morphism* $\mathrm{Spec}\,C\to Y$ *(which we use to view* $\mathrm{Spec}\,C$ *and* $\mathrm{Spec}\,C/I$ *as Y-schemes), the composition with the natural closed embedding* $\mathrm{Spec}\,C/I\to\mathrm{Spec}\,C$ *yields an injective map* $\mathrm{Hom}_Y(\mathrm{Spec}\,C,X)\to\mathrm{Hom}_Y(\mathrm{Spec}\,C/I,X)$.

(2) *We say that f is* formally smooth, *if for every ring C, every ideal I with $I^2 = 0$, and every morphism* $\operatorname{Spec} C \to Y$*, the composition with the natural closed embedding* $\operatorname{Spec} C/I \to \operatorname{Spec} C$ *yields a surjective map* $\operatorname{Hom}_Y(\operatorname{Spec} C, X) \to \operatorname{Hom}_Y(\operatorname{Spec} C/I, X)$.
(3) *We say that f is* formally Ã©tale, *if f is formally unramified and formally smooth.*

If f is a morphism of affine schemes, then f has one of the properties of this definition if and only if the corresponding ring homomorphism has the same property in the sense of our previous definition.

Lemma 3.33.
(1) *Every monomorphism of schemes (in particular: every immersion) is formally unramified.*
(2) *Let $A \to B \to C$ be ring homomorphisms such that $A \to B$ is formally unramified. Then we can naturally identify $\Omega_{C/A} = \Omega_{C/B}$.*

Definition 3.34. *Let $X \to Y$ be a morphism of schemes, and let $\mathscr{M}$ be an $\mathscr{O}_X$-module. A derivation $D\colon \mathscr{O}_X \to \mathscr{M}$ is a homomorphism of abelian sheaves such that for all open subsets $U \subseteq X$, $V \subseteq Y$ with $f(U) \subseteq V$, the map $\mathscr{O}(U) \to \mathscr{M}(U)$ is an $\mathscr{O}_Y(V)$-derivation.*

Equivalently, $D\colon \mathscr{O}_X \to \mathscr{M}$ is a homomorphism of $f^{-1}(\mathscr{O}_Y)$-modules such that for every open $U \subseteq X$, the Leibniz rule

$$D(U)(bb') = bD(U)(b') + b'D(U)(b), \qquad \forall b, b' \in \Gamma(U, \mathscr{O}_X)$$

holds.

We denote the set of all these derivations by $\operatorname{Der}_Y(\mathscr{O}_X, \mathscr{M})$*; it is a* $\Gamma(X, \mathscr{O}_X)$*-module.*

May 22, 2019

Definition/Proposition 3.35. *Let $f\colon X \to Y$ be a morphism of schemes. The following three definitions give the same result (up to unique isomorphism), called the sheaf of differentials of f or of X over Y, denoted $\Omega_{X/Y}$ — a quasi-coherent $\mathscr{O}_X$-module together with a derivation $d_{X/Y}\colon \mathscr{O}_X \to \Omega_{X/Y}$.*

(i) *There exists a unique $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with a derivation $d_{X/Y}\colon \mathscr{O}_X \to \Omega_{X/Y}$ such that for all affine open subsets* $\operatorname{Spec} B = U \subseteq X$, $\operatorname{Spec} A = V \subseteq Y$ *with $f(U) \subseteq V$, $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$ and $d_{X/Y|U}$ is induced by $d_{B/A}$.*
(ii) $\Omega_{X/Y} = \Delta^*(\mathscr{J}/\mathscr{J}^2)$, *where $\Delta\colon X \to X \times_Y X$ is the diagonal morphism, $W \subseteq X \times_Y X$ is open such that $\operatorname{im}(\Delta) \subseteq W$ is closed (if f is separated we can take $W = X \times_Y X$), and $\mathscr{J}$ is the quasi-coherent ideal defining the closed subscheme $\Delta(X) \subseteq W$. The derivation $d_{X/Y}$ is induced, on affine opens, by the map $b \mapsto 1 \otimes b - b \otimes 1$.*
(iii) *The quasi-coherent $\mathscr{O}_X$-module $\Omega_{X/Y}$ together with $d_{X/Y}$ is characterized by the universal property that composition with $d_{X/Y}$ induces bijections*

$$\operatorname{Hom}_{\mathscr{O}_X}(\Omega_{X/Y}, \mathscr{M}) \xrightarrow{\sim} \operatorname{Der}_Y(\mathscr{O}_X, \mathscr{M})$$

for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$, functorially in $\mathscr{M}$.

The properties we proved for modules of differentials can be translated into statements for sheaves of differentials:

Proposition 3.36. *Let $f\colon X \to Y$, $g\colon Y' \to Y$ be morphisms of schemes, and let $X' = X \times_Y Y'$. Denote by $g'\colon X' \to X$ the base change of g. There is a natural isomorphism $\Omega_{X'/Y'} \cong (g')^*\Omega_{X/Y}$, compatible with the universal derivations.*

Proposition 3.37. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms of schemes. Then there is an exact sequence*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

of $\mathscr{O}_X$-modules. If f is formally smooth, then the sequence

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

is exact and splits locally on X.

Proposition 3.38. *Let $i\colon Z \to X$ be a closed immersion with corresponding ideal sheaf $\mathscr{J} \subseteq \mathscr{O}_X$, and let $g\colon X \to Y$ be a scheme morphism. Then there is an exact sequence*

$$i^*(\mathscr{J}/\mathscr{J}^2) \to i^*\Omega_{X/Y} \to \Omega_{Z/Y} \to 0$$

of $\mathscr{O}_Z$-modules. If Z is formally smooth over Y, then the sequence

$$0 \to i^*(\mathscr{J}/\mathscr{J}^2) \to i^*\Omega_{X/Y} \to \Omega_{Z/Y} \to 0$$

is exact and splits locally on Z.

Proposition 3.39. *Let K be a field, and let X be a k-scheme of finite type. Let $x \in X(k)$. Then we have an isomorphism $T_xX = \Omega_{X/k}(x)$ between the Zariski tangent space at x and the fiber of the sheaf of differentials of X/k at x.*

(3.9) Sheaves of differentials and smoothness.

We start by slightly rephrasing the definition of a smooth morphism.

Definition 3.40. *A morphism $f\colon X \to Y$ of schemes is called* smooth of relative dimension $d \geq 0$ in $x \in X$, *if there exist affine open neighborhoods $U \subseteq X$ of x and $V = \operatorname{Spec} R \subseteq Y$ of $f(x)$ such that $f(U) \subseteq V$ and an open immersion $j\colon U \to \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d})$ such that the triangle*

$$\begin{array}{ccc} U & \xrightarrow{\ j\ } & \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d}) \\ & {\scriptstyle f}\searrow & \swarrow \\ & V & \end{array}$$

is commutative, and that the images of $df_1, \dots, df_{n-d}$ in the fiber $\Omega^1_{\mathbb{A}^n_R/R} \otimes \kappa(x)$ are linearly independent over $\kappa(x)$. (We view x as a point of $\mathbb{A}^n_R$ via *the embedding* $U \to \operatorname{Spec} R[T_1, \dots, T_n](f_1, \dots, f_{n-d}) \to \operatorname{Spec} R[T_1, \dots, T_n] = \mathbb{A}^n_R$*.)*

Proposition 3.41. *Let* $f\colon X \to S$ *be smooth of relative dimension at* $x \in X$*. Then there exists an open neighborhood* U *of* x *such that the restriction* $\Omega_{X/Y|U} (= \Omega_{U/Y})$ *is free of rank* d*.*

May 27, 2019

Theorem 3.42. *Let* k *be an algebraically closed field, and let* X *be an irreducible* k*-scheme of finite type. Let* $d = \dim X$*. Then* X *is smooth over* k *if and only if* $\Omega_{X/k}$ *is locally free of rank* d*.*

Proposition 3.43. *Let* $f\colon X \to S$ *be smooth of relative dimension* d *at* $x \in X$*. Then there exists an open neighborhood* U *of* x *such that the restriction* $U \to S$ *of* f *to* U *is formally smooth.*

Theorem 3.44. *Let* $f\colon X \to Y$ *be a morphism locally of finite presentation (e.g., if* Y *is noetherian and* f *is locally of finite type). Then* f *is smooth if and only if* f *is formally smooth.*

We skip the proof that smoothness implies formal smoothness, see for instance [Bo] Ch. 8.5. (But cf. the previous proposition which shows that a smooth morphism is at least "locally formally smooth".)

May 29, 2019

Projective schemes.

References: [GW], Ch. 8, Ch. 11, in particular Example 11.43, (8.5); [H] II.6, II.7.

(3.10) Line bundles on $\mathbb{P}^n_k$.

Let R be a ring. We cover $\mathbb{P}^n_R$ by the standard charts $U_i := D_+(T_i)$, as usual, and write $U_{ij} := U_i \cap U_j$. For $d \in \mathbb{Z}$, the elements $(T_i/T_j)^d \in \Gamma(U_{ij}, \mathscr{O}_{\mathbb{P}^n_R})^\times$ define isomorphisms $\mathscr{O}_{U_i|U_{ij}} \to \mathscr{O}_{U_j|U_{ij}}$ which give rise to a gluing datum of the $\mathscr{O}_{U_i}$-modules $\mathscr{O}_{U_i}$. By gluing of sheaves, we obtain a line bundle $\mathscr{O}_{\mathbb{P}^n_R}(d)$. (Cf. Problems 9, 10.)

Lemma 3.45. *We obtain a group homomorphism* $\mathbb{Z} \to \operatorname{Pic}(\mathbb{P}^n_R)$*,* $d \mapsto \mathscr{O}(d)$*.*

Proposition 3.46. *Writing* $R[T_0, \dots, T_n]_d$ *for the submodule of homogeneous polynomials of degree* d *(with* $R[T_0, \dots, T_n]_d = 0$ *for* $d < 0$*), we have*

$$\Gamma(\mathbb{P}^n_R, \mathscr{O}(d)) \cong R[T_0, \dots, T_n]_d$$

for all $d \in \mathbb{Z}$*.*

Now let $R = k$ be a field.

The closed subscheme $V_+(T_0)$ is a Weil divisor on $\mathbb{P}^n_k$, and it corresponds to the Cartier divisor $(U_i, T_0/T_i)_i$. The corresponding line bundle is $\mathscr{O}(1)$. By passing to multiples/negatives of this divisor, we can describe all $\mathscr{O}(d)$ in a similar way.

June 3, 2019

Remark 3.47. One can show that every locally free $\mathscr{O}_{\mathbb{P}^1_k}$-module is isomorphic to a direct sum of line bundles. Note though that this statement is not true for $\mathbb{P}^n_k$, $n > 1$.

Proposition 3.48. *Let A be a unique factorization domain, and let $Z = V(\mathfrak{p}) \subset \operatorname{Spec} A$ a closed irreducible subset of codimension 1, i.e., $\mathfrak{p} \neq 0$ is a prime ideal which is minimal among all non-zero prime ideals. Then $\mathfrak{p}$ is a principal ideal, i.e., considering Z as a Weil divisor, it is principal.*

Corollary 3.49. *Let k be a field, and let $Z \subset \mathbb{A}^n_k$ be an integral closed subscheme of codimension 1. Then $Z = V(f)$ for some polynomial f.*

Proposition 3.50. *Let k be a field, and let $Z \subseteq \mathbb{P}^n_k$ be an integral closed subscheme of codimension 1. Then $Z = V_+(f)$ for some homogeneous polynomial f.*

Proposition 3.51. *The above homomorphism $\mathbb{Z} \to \operatorname{Pic}(\mathbb{P}^n_k)$, $d \mapsto \mathscr{O}(d)$, is an isomorphism.*

Proposition 3.52. *Let R be a ring. We have a short exact sequence*

$$0 \to \Omega_{\mathbb{P}^n_R/R} \to \mathscr{O}(-1)^{n+1} \to \mathscr{O}^n \to 0$$

of $\mathscr{O}_X$-modules.

(3.11) Functorial description of $\mathbb{P}^n$.

June 5, 2019

As we have seen last term, every scheme X defines a contravariant functor $T \mapsto X(T) := \operatorname{Hom}_{(\mathrm{Sch})}(T, X)$ from the category of schemes to the category of sets. This functor determines X up to unique isomorphism. In this section, we want to describe the functor attached in this way to projective space $\mathbb{P}^n_R$ for R a ring.

Proposition 3.53. *Let R be a ring, and let S be an R-scheme. There is a bijection, functorial in S,*

$$\mathbb{P}^n_R(S) = \{(\mathscr{L}, \alpha);\ \mathscr{L} \text{ a line bundle on } S,\quad \alpha\colon \mathscr{O}_S^{n+1} \twoheadrightarrow \mathscr{L} \text{ a surjective} \mathscr{O}_S\text{-module homom.}\}/\cong$$

Here we consider pairs $(\mathscr{L}, \alpha)$, $(\mathscr{L}', \alpha')$ as isomorphic, if there exists an $\mathscr{O}_S$-module isomorphism $\beta\colon \mathscr{L} \to \mathscr{L}'$ with $\alpha = \alpha' \circ \beta$.

Note that a homomorphism $\alpha\colon \mathscr{O}_S^{n+1} \twoheadrightarrow \mathscr{L}$ corresponds to $n+1$ global sections in $\Gamma(S, \mathscr{L})$ (the "images of the standard basis vectors"). Thus $T_0, \dots, T_n \in \Gamma(\mathbb{P}^n_R, \mathscr{O}(1))$ give rise to a (surjective) homomorphism $\mathscr{O}_{\mathbb{P}^n_R}^{n+1} \to$

$\mathscr{O}(1)$. Given a morphism $S \to \mathbb{P}^n_R$, we can pull this homomorphism back to S and obtain an element of the right hand side in the statement of the proposition.

Conversely, given a pair $(\mathscr{L}, \alpha)$ on S, we can think of the corresponding morphism $S \to \mathbb{P}^n_R$ in terms of homogeneous coordinates (i.e., for K-valued points for some field K), as follows: Denote by $f_0, \to f_n \in \Gamma(S, \mathscr{L})$ the global sections corresponding to α. For a point $x \in S$, the fiber $\mathscr{L}(x)$ is a one-dimensional $\kappa(x)$-vector space generated by the elements $f_0(x), \dots, f_n(x)$ (i.e., at least one of them ins $\neq 0$ – this holds since α is surjective). We choose an isomorphism $\mathscr{L}(x) \cong \kappa(x)$, and hence can view the $f_i(x)$ as elements of $\kappa(x)$. Then the morphism $S \to \mathbb{P}^n_S$ maps x to $(f_0(x) : \cdots : f_n(x)) \in \mathbb{P}^n(\kappa(x))$. While the individual $f_i(x)$, as elements of $\kappa(x)$, depend on the choice of isomorphism $\mathscr{L}(x) \cong \kappa(x)$, the point $(f_0(x) : \cdots : f_n(x)) \in \mathbb{P}^n(\kappa(x))$ is independent of this choice.

(3.12) The Proj construction.

Reference: [GW] Ch. 13.

Definition 3.54.

(1) *A* graded ring *is a ring A with a decomposition $A = \bigoplus_{d \geq 0} A_d$ as abelian groups such that $A_d \cdot A_e \subseteq A_{d+e}$ for all d, e. The elements of A_d are called* homogeneous of degree d.

(2) *Let R be a ring. A* graded R-algebra *is a graded ring A together with a ring homomorphism $R \to A$.*

(3) *A* homomorphism $A \to B$ of graded rings (or graded R-algebras) *is a ring homomorphism (or R-algebra homomorphism, respectively) $f\colon A \to B$ such that $f(A_d) \subseteq B_d$ for all d.*

(4) *Let A be a graded ring. A* graded A-module *is an A-module M with a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ such that $A_d \cdot M_e \subseteq M_{d+e}$ for all d, e. The elements of M_d are called* homogeneous of degree d.

(5) *A* homomorphism $M \to N$ of graded A-modules *is an A-module homomorphism $f\colon M \to N$ such that $f(M_d) \subseteq N_d$ for all d.*

(6) *Let A be a graded ring and let M be a graded A-module. A* homogeneous submodule of M *is a submodule $N \subseteq M$ such that $N = \bigoplus_{d \in \mathbb{Z}} (N \cap M_d)$. In this way, N is itself a graded A-module and the inclusion $N \hookrightarrow M$ is a homomorphism of graded A-modules. (And conversely, every injective homomorphism of graded A-modules has a homogeneous submodule as its image.) A homogeneous submodule of A is called a* homogeneous ideal.

Example 3.55. Let R be a ring. Then the polynomial ring $R[T_0, \dots, T_n]$ is a graded R-algebra if we set $R[T_0, \dots, T_n]_d$ to be the R-submodule of homogeneous polynomials of degree d.

We now fix a graded ring A.

For a homogeneous element $f \in A_e$, and a graded A-module M, the localization M_f is a graded A-module via

$$M_{f,d} = \{\frac{m}{f^i};\ m \in M_{d+ei}\}.$$

Applying this to A as an A-module, we obtain a grading on A_f giving A_f the structure of a graded ring. Then M_f is a graded A_f-module.

We define

$$M_{(f)} := M_{f,0},$$

the degree 0 part of M_f. Then $A_{(f)}$ is a ring and $M_{(f)}$ is an $A_{(f)}$-module.

Example 3.56. Let R be a ring. Then

$$R[T_0, \dots, T_n]_{(T_i)} = R[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}].$$

Definition 3.57. *We write $A_+ := \bigoplus_{d>0} A_d$, an ideal of A. A homogeneous prime ideal $\mathfrak{p} \subset A$ is called* relevant *if $A_+ \not\subseteq \mathfrak{p}$.*

Definition 3.58. *We denote by* $\operatorname{Proj}(A)$ *the set of all relevant homogeneous prime ideals of A. We equip* $\operatorname{Proj}(A)$ *with the Zariski topology, by saying that the closed subsets are the subsets of the form*

$$V_+(I) := \{\mathfrak{p} \in \operatorname{Proj}(A);\ I \subseteq \mathfrak{p}\}.$$

for homogeneous ideals $I \subseteq A$.

For a homogeneous element f, we write $D_+(f) := \operatorname{Proj}(A) \setminus V_+(f)$.

Lemma 3.59. *Let $f \in A$ be a homogeneous element. Then the map*

$$D_+(f) \to \operatorname{Spec} A_{(f)}, \quad \mathfrak{p} \mapsto (\mathfrak{p}A_f) \cap A_{(f)}$$

is a homeomorphism.

Proposition 3.60. *There is a unique sheaf $\mathscr{O}$ of rings on* $\operatorname{Proj}(A)$ *such that*

$$\Gamma(D_+(f), \mathscr{O}) = A_{(f)}$$

for every homogeneous element $f \in A$ and with restriction maps given by the canonical maps between the localizations. The ringed space $(\operatorname{Proj}(A), \mathscr{O})$ *is a separated scheme which we again denote by* $\operatorname{Proj}(A)$.

June 17, 2019

Definition 3.61. *Let R be a ring, and let X be an R-scheme. We say that X is* projective over R *(or that the morphism $X \to \operatorname{Spec} R$ is* projective*), if there exist $n \geq 0$ and a closed immersion $X \to \mathbb{P}^n_R$ of R-schemes.*

Theorem 3.62. *Let R be a ring, and let X be a projective R-scheme. Then X is proper over R.*

(3.13) Quasi-coherent modules on $\operatorname{Proj}(A)$**.**

Let A be a graded ring, $X = \operatorname{Proj} A$. If M is a graded A-module, there is a unique sheaf $\widetilde{M}$ of $\mathscr{O}_X$-modules such that

$$\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$$

for every homogeneous element $f \in A$, and such that the restriction maps for inclusions of the form $D_+(g) \subseteq D_+(f)$ are given by the natural maps between the localizations. This sheaf is a quasi-coherent $\mathscr{O}_X$-module.

Example 3.63. Let $A(n)$ be the graded A-module defined by $A(n) = \bigoplus_{d\in\mathbb{Z}} A_{n+d}$. We set $\mathscr{O}_X(n) = \widetilde{A(n)}$. If $A = R[T_0, \dots, T_n]$ for a ring R, so that $X = \mathbb{P}^n_R$, then this notation is consistent with our previous definition.

For $f \in A_d$, multiplication by f^k defines an isomorphism

$$\mathscr{O}_{X|D_+(f)} \xrightarrow{\sim} \mathscr{O}_X(n)_{|D_+(f)}.$$

In particular, if A is generated as an A_0-algebra by A_1, then $\mathscr{O}_X(n)$ is a line bundle.

Assume, for the remainder of this section, that A is generated as an A_0-algebra by A_1. So $X = \bigcup_{f\in A_1} D_+(f)$, and $\mathscr{O}_X(n)$ is a line bundle.

For an $\mathscr{O}_X$-module $\mathscr{F}$, write $\mathscr{F}(n) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$, and define a graded A-module $\Gamma_*(\mathscr{F})$ by

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathscr{F}(n)).$$

Lemma 3.64. *For a graded A-module M, there is a natural map $M \to \Gamma_*(\widetilde{M})$. For an $\mathscr{O}_X$-module $\mathscr{F}$, there is a natural map $\widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$. If $\mathscr{F}$ is quasi-coherent, then the latter map is an isomorphism.*

Call a graded A-module M *saturated*, if the map $M \to \Gamma_*(\widetilde{M})$ is an isomorphism.

Proposition 3.65. *The functors $M \to \widetilde{M}$ and $\mathscr{F} \to \Gamma_*(\mathscr{F})$ define an equivalence of categories between the category of saturated graded A-modules and the category of quasi-coherent $\mathscr{O}_X$-modules.*

4. Cohomology of $\mathscr{O}_X$-modules

June 19, 2019

General references: [We], [HS], [Gr], [KS].

4.1. The formalism of derived functors.

(4.1) Complexes in abelian categories.

Reference: [We] Ch. 1.

Let $\mathcal{A}$ be an abelian category (e.g., the category of abelian groups, the category of R-modules for a ring R, the category of abelian sheaves on a topological space X, the category of $\mathscr{O}_X$-modules on a ringed space X, or the category of quasi-coherent $\mathscr{O}_X$-modules on a scheme X).

A *complex in* $\mathcal{A}$ is a sequence of morphisms

$$\cdots \longrightarrow A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \longrightarrow \cdots$$

in $\mathcal{A}$ ($i \in \mathbb{Z}$), such that $d^{i+1} \circ d^i = 0$ for every $i \in \mathbb{Z}$. The maps d^i are called the *differentials* of the complex.

Given complexes $A^\bullet$, $B^\bullet$, a morphism $A^\bullet \to B^\bullet$ of complexes is a family of morphisms $f^i\colon A^i \to B^i$ such that the f^i commute with the differentials of $A^\bullet$ and $B^\bullet$ in the obvious way. With this notion of morphisms, we obtain the category $C(\mathcal{A})$ of complexes in $\mathcal{A}$. This is an abelian category (kernels, images, …are formed degree-wise); see [We] Thm. 1.2.3.

Definition 4.1. *Let $A^\bullet$ be a complex in $\mathcal{A}$. For $i \in \mathbb{Z}$, we call*

$$h^i(A^\bullet) := \operatorname{Ker}(d^i)/\operatorname{im}(d^{i-1})$$

the i-th cohomology object *of $A^\bullet$. We obtain functors $h^i\colon C(\mathcal{A}) \to \mathcal{A}$. We say that $A^\bullet$ is* exact at i*, if $h^i(A^\bullet) = 0$. We say that $A^\bullet$ is* exact*, if $h^i(A^\bullet) = 0$ for all i.*

Remark 4.2. Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a sequence of morphisms of complexes. The sequence is exact (in the sense that at each point the kernel and image in the category $C(\mathcal{A})$ coincide) if and only if for every i, the sequence $0 \to A^i \to B^i \to C^i \to 0$ is exact.

Proposition 4.3. *Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be an exact sequence of complexes in $\mathcal{A}$. Then there are maps $\delta^i\colon h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ (called* boundary maps*) such that, together with the maps induced by functoriality of the h^i, we obtain the* long exact cohomology sequence

$$\cdots h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \to h^{i+1}(A^\bullet) \to \cdots .$$

Reference: [We] *Thm. 1.3.1.*

We need a criterion which ensures that two morphisms between complexes induce the same maps on all cohomology objects. Reference: [We] 1.4.

Definition 4.4. *Let $f, g\colon A^\bullet \to B^\bullet$ be morphisms of complexes. We say that f and g are* homotopic*, if there exists a family of maps $k^i\colon A^i \to B^{i-1}$, $i \in \mathbb{Z}$, such that*

$$f - g = dk + kd,$$

which we use as short-hand notation for saying that for every i,

$$f^i - g^i = d_B^{i-1} \circ k^i + k^{i+1} \circ d_A^i.$$

In this case we write $f \sim g$. The family $(k^i)_i$ is called a homotopy.

Proposition 4.5. *Let $f, g\colon A^\bullet \to B^\bullet$ be morphisms of complexes which are homotopic. Then for every i, the maps $h^i(A^\bullet) \to h^i(B^\bullet)$ induced by f and g are equal.*

In particular, if $A^\bullet$ is a complex such that $\mathrm{id}_{A^\bullet} \sim 0$, then $h^i(A^\bullet) = 0$ for all i, i.e., $A^\bullet$ is exact.

Definition 4.6. *Let $A^\bullet$ and $B^\bullet$ be complexes. We say that $A^\bullet$ and $B^\bullet$ are* homotopy equivalent*, if there exist morphisms $f\colon A^\bullet \to B^\bullet$ and $g\colon B^\bullet \to A^\bullet$ of complexes such that $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_B$. In this case, f and g induce isomorphisms $h^i(A^\bullet) \cong h^i(B^\bullet)$ for all i.*

June 24, 2019

(4.2) Left exact functors.

Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. All functors $F\colon \mathcal{A} \to \mathcal{B}$ that we consider are assumed to be additive, i.e., they induce group homomorphisms $\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(A'))$ for all A, A' in $\mathcal{A}$.

Definition 4.7. *A (covariant) functor $F\colon \mathcal{A} \to \mathcal{B}$ is called* left exact*, if for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, the sequence*

$$0 \to F(A') \to F(A) \to F(A'')$$

is exact.

Definition 4.8. *A contravariant functor $F\colon \mathcal{A} \to \mathcal{B}$ is called* left exact *if for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, the sequence*

$$0 \to F(A'') \to F(A) \to F(A')$$

is exact.

Similarly, we have the notion of right exact functor. A functor which is left exact and right exact (and hence preserves exactness of arbitrary sequences) is called exact.

Let $A_0 \in \mathcal{A}$. Then the functors $A \mapsto \mathrm{Hom}_{\mathcal{A}}(A, A_0)$ and $A \mapsto \mathrm{Hom}_{\mathcal{A}}(A_0, A)$ are left exact.

(4.3) δ-functors.

Reference: [We] 2.1.

Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories.

Definition 4.9. *A* δ-functor *from $\mathcal{A}$ to $\mathcal{B}$ is a family $(T^i)_{i\geq 0}$ of functors $\mathcal{A} \to \mathcal{B}$ together with morphisms $\delta^i\colon T^i(A'') \to T^{i+1}(A')$ (called* boundary morphisms*) for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, such that the sequence*

$$0 \to T^0(A') \to T^0(A) \to T^0(A'') \to T(A') \to \cdots$$

is exact, and such that the δ^i are compatible with morphisms of short exact sequences in the obvious way.

Definition 4.10. *A δ-functor $(T^i)_i$ from $\mathcal{A}$ to $\mathcal{B}$ is called* universal*, if for every δ-functor $(S^i)_i$ and every morphism $f^0\colon T^0 \to S^0$ of functors, there exist unique morphisms $f^i\colon T^i \to S^i$ of functors for al $i > 0$, such that the f^i, $i \geq 0$ are compatible with the boundary maps δ^i of the two δ-functors for each short exact sequence in $\mathcal{A}$.*

The definition implies that given a (left exact) functor F, any two universal δ-functors $(T^i)_i$, $(T'^i)_i$ with $T^0 = T'^0 = F$ are isomorphic (in the obvious sense) via a unique isomorphism.

Definition 4.11. *A functor $F\colon \mathcal{A} \to \mathcal{B}$ is called* effaceable*, if for every X in $\mathcal{A}$ there exists a monomorphism $\iota\colon X \hookrightarrow A$ with $F(\iota) = 0$.*

A particular case is the situation where each X admits a monomorphism to an object I with $F(I) = 0$.

Proposition 4.12. *Let $(T^i)_i$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$ such that for every $i > 0$, the functor T^i is effaceable. Then $(T^i)_i$ is a universal δ-functor.*
Reference: [We] *Thm. 2.4.7, Ex. 2.4.5.*

(4.4) Injective objects.

Let $\mathcal{A}$ be an abelian category.

Definition 4.13. *An object I in $\mathcal{A}$ is called* injective*, if the functor $X \mapsto \mathrm{Hom}_{\mathcal{A}}(X, I)$ is exact.*

If I is injective, then every short exact sequence $0 \to I \to A \to A'' \to 0$ in $\mathcal{A}$ splits. Conversely, if I is an object such that every short exact sequence $0 \to I \to A \to A'' \to 0$ splits, then I is injective.

Definition 4.14. *Let $X \in \mathcal{A}$. An* injective resolution *of X is an exact sequence*

$$0 \to X \to I^0 \to I^1 \to I^2 \to \cdots$$

in $\mathcal{A}$, where every I^i is injective.

Definition 4.15. *We say that the category $\mathcal{A}$* has enough injectives *if for every object X there exists a monomorphism $X \hookrightarrow I$ from X into an injective object I. Equivalently: Every object has an injective resolution.*

The categories of abelian groups, of R-modules for a ring R, of abelian sheaves on a topological space, and more generally of $\mathscr{O}_X$-modules on a ringed space X all have enough injective objects.

Dually, we have the notion of *projective* object (i.e., P such that $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact), of *projective resolution* $\cdots \to P_1 \to P_0 \to A \to 0$, and of abelian categories with *enough projective objects*. For a ring R, the category of R-modules clearly has enough projectives, since every free module is projective, and every module admits an epimorphism from a free module. Categories of sheaves of abelian groups or $\mathscr{O}_X$-modules typically do not have enough projectives.

(4.5) Right derived functors.

June 26, 2019

Theorem 4.16. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor, and assume that $\mathcal{A}$ has enough injectives.*

For each $A \in \mathcal{A}$, fix an injective resolution $0 \to A \to I^\bullet$, and define

$$R^iF(A) = h^i(F(I^\bullet)), \qquad i \geq 0,$$

where $F(I^\bullet)$ denotes the complex obtained by applying the functor to all I^i and to the differentials of the complex $I^b ullet$. Then:

(1) *The R^iF are additive functors $\mathcal{A} \to \mathcal{B}$, and R^iFX is independent of the choice of injective resolution of X up to natural isomorphism.*
(2) *We have an isomorphism $F \cong R^0F$ of functors.*
(3) *For I injective, we have $R^iFI = 0$ for all $i > 0$.*
(4) *The family $(R^iF)_i$ is a universal δ-functor.*

We call the R^iF the right derived functors *of F.*

Definition 4.17. *Let F be a left exact functor as above. We say that an object $A \in \mathcal{A}$ is F-*acyclic*, if $R^iF(A) = 0$ for all $i > 0$.*

Definition 4.18. *Let F be a left exact functor as above, and let $A \in \mathcal{A}$. An F-*acyclic resolution *of A is an exact sequence $0 \to A \to J^0 \to J^1 \to \cdots$ where all J^i are F-acyclic.*

Proposition 4.19. *Let F be a left exact functor as above, and let $A \in \mathcal{A}$. Let $0 \to A \to J^0 \to J^1 \to \cdots$ be an F-acyclic resolution. Then we have natural isomorphisms $R^iF(A) = h^i(F(J^\bullet))$, i.e., we can compute $R^iF(A)$ by an F-acyclic resolution.*

4.2. **Cohomology of sheaves.**

General reference: [H] Ch. III, [Stacks] Ch. 20, 29.

(4.6) Cohomology groups.

Let X be a topological space. Denote by (Ab_X) the category of abelian sheaves (i.e., sheaves of abelian groups) on X. We have the global section functor

$$\Gamma\colon (\mathrm{Ab}_X) \to (\mathrm{Ab}), \quad \mathscr{F} \mapsto \Gamma(X, \mathscr{F}),$$

to the category of abelian groups. This is a left exact functor, and we denote its right derived functors by $H^i(X,-)$. We call $H^i(X,\mathscr{F})$ the *i-th cohomology group of X with coefficients in* $\mathscr{F}$.

Example 4.20. For a field k, $H^1(\mathbb{P}^1_k,\mathscr{O}(-2))\neq 0$.

July 1, 2019

(4.7) Flasque sheaves.

Definition 4.21. *Let X be a topological space. A sheaf $\mathscr{F}$ on X is called* flasque *(or* flabby*), if all restriction maps $\mathscr{F}(U)\to\mathscr{F}(V)$ for $V\subseteq U\subseteq X$ open are surjective.*

Lemma 4.22. *Let X be a ringed space. Let $\mathscr{F}$ be an injective object in the category of $\mathscr{O}_X$-modules. Then $\mathscr{F}$ is flasque.*

Proposition 4.23. *Let X be a topological space, and let $\mathscr{F}$ be a flasque abelian sheaf on X. Then $\mathscr{F}$ is Γ-acyclic, i.e., $H^i(X,\mathscr{F})=0$ for all $i>0$.*

Corollary 4.24. *Let X be a ringed space. The right derived functors of the global section functor from the category of $\mathscr{O}_X$-modules to the category of abelian groups can naturally be identified with $H^i(X,-)$.*

It follows that for an $\mathscr{O}_X$-module $\mathscr{F}$ the cohomology groups $H^i(X,\mathscr{F})$ carry a natural $\Gamma(X,\mathscr{O}_X)$-module structure.

(4.8) Grothendieck vanishing.

Reference: [H] III.2.

Lemma 4.25. *Let X be a topological space, and let $\iota\colon Y\to X$ be the inclusion of a closed subset Y of X. Let $\mathscr{F}$ be an abelian sheaf on X. Then there are natural isomorphisms*

$$H^i(Y,\mathscr{F})=H^i(X,\iota_*\mathscr{F}),\quad i\geq 0.$$

Theorem 4.26. (Grothendieck) *Let X be a noetherian topological space (i.e., the descending chain condition holds for closed subsets of X), let $n=\dim X$, and let $\mathscr{F}$ be a sheaf of abelian groups on X. Then*

$$H^i(X,\mathscr{F})=0\quad\text{for all } i>n.$$

July 3, 2019

4.3. **Čech cohomology.**

Reference: [H] III.4, [Stacks] `01ED` (and following sections); a classical reference is [Go].

(4.9) Čech cohomology groups.

Let X be a topological space, and let $\mathscr{F}$ be an abelian sheaf on X. (The definitions below can be made more generally for presheaves.)

Let $\mathscr{U} = (U_i)_{i\in I}$ be an open cover of X. We fix a total order of the index set I (but see below for a sketch that the results are independent of this). For $i_0, \dots, i_p \in I$, we write $U_{i_0\dots i_p} := \bigcap_{\nu=0}^p U_{i_\nu}$.

We define

$$C^p(\mathscr{U}, \mathscr{F}) = \prod_{i_0<\dots<i_p} \Gamma(U_{i_0\dots i_p}, \mathscr{F})$$

and

$$d\colon C^p(\mathscr{U}, \mathscr{F}) \to C^{p+1}(\mathscr{U}, \mathscr{F}), \quad (s_{\underline{i}})_{\underline{i}} \mapsto \left(\sum_{\nu=0}^{p+1} (-1)^\nu s_{i_0\dots\widehat{i_\nu}\dots i_p|U_{\underline{i}}}\right)_{\underline{i}},$$

where $\widehat{\cdot}$ indicates that the corresponding index is omitted. One checks that $d \circ d = 0$, so we obtain a complex, the so-called *Čech complex for the cover $\mathscr{U}$ with coefficients in $\mathscr{F}$.*

Definition 4.27. *The* Čech cohomology groups *for $\mathscr{U}$ with coefficients in $\mathscr{F}$ are defined as*

$$\check{H}^p(\mathscr{U}, \mathscr{F}) = h^p(C^\bullet(\mathscr{U}, \mathscr{F})), \quad p \geq 0.$$

Since $\mathscr{F}$ is a sheaf, we have $\check{H}^0(\mathscr{U}, \mathscr{F}) = \Gamma(X, \mathscr{F}) = H^0(X, \mathscr{F})$.

(4.10) The "full" Čech complex.

Instead of the "alternating" (or "ordered") Čech complex as above, we can also consider the "full" Čech complex

$$C_f^p(\mathscr{U}, \mathscr{F}) = \prod_{i_0,\dots,i_p} \Gamma(U_{i_0\dots i_p}, \mathscr{F}),$$

with differentials defined by the same formula as above. Then the projection $C_f^\bullet(\mathscr{U}, \mathscr{F}) \to C^\bullet(\mathscr{U}, \mathscr{F})$ is a homotopy equivalence, with "homotopy inverse" given by

$$(s_{\underline{i}})_{\underline{i}} \mapsto (t_{\underline{i}})_{\underline{i}},$$

where $t_{\underline{i}} = 0$ whenever two entries in the multi-index $\underline{i}$ coincide, and otherwise $t_{\underline{i}} = \mathrm{sgn}(\sigma) s_{\sigma(\underline{i})}$, where σ is the permutation such that $\sigma(\underline{i})$ is in increasing order.

In particular, we have natural isomorphisms between the cohomology groups of the two complexes. So we also see that the Čech cohomology groups as defined above are independent of the choice of order on I.

(4.11) Passing to refinements.

Definition 4.28. *A* refinement *of a cover* $\mathscr{U} = (U_i)_i$ *of* X *is a cover* $\mathscr{V} = (V_j)_{j\in J}$ *(with* J *totally ordered) together with a map* $\lambda\colon J \to I$ *respecting the orders on* I *and* J *such that* $V_j \subseteq U_{\lambda(j)}$ *for every* $j \in J$.

Given a refinement $\mathscr{V}$ of $\mathscr{U}$, one obtains a natural map (using restriction of sections to smaller open subsets)

$$\check{H}^p(\mathscr{U}, \mathscr{F}) \to \check{H}^p(\mathscr{V}, \mathscr{F}).$$

We can pass to the colimit over all these maps given by refinements, and define

$$\check{H}^p(X, \mathscr{F}) := \operatorname*{colim}_{\mathscr{U}} \check{H}^p(\mathscr{U}, \mathscr{F}),$$

the *p-th Čech cohomology group of* X *with coefficients in* $\mathscr{F}$.

Proposition 4.29. *Let* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *be a short exact sequence of abelian sheaves on* X*. Then there exists a homomorphism* $\delta\colon \Gamma(X, \mathscr{F}'') \to \check{H}^1(X, \mathscr{F})$ *such that the sequence*

$$\begin{aligned} 0 \to &\Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \\ &\to \check{H}^1(X, \mathscr{F}') \to \check{H}^1(X, \mathscr{F}) \to \check{H}^1(X, \mathscr{F}'') \end{aligned}$$

is exact. (But note that the sequence does not continue after $\check{H}^1(X, \mathscr{F}'')$*.)*

(4.12) A sheaf version of the Čech complex.

We define a sheaf version of the Čech complex as follows:

$$\mathscr{C}^p(\mathscr{U}, \mathscr{F}) = \prod_{\underline{i}=(i_0<\cdots<i_p)} j_{\underline{i},*}(\mathscr{F}_{|U_{\underline{i}}}),$$

with differentials defined by (basically) the same formula as above. Here $j_{\underline{i}}$ denotes the inclusion $U_{\underline{i}} \hookrightarrow X$.

We have a natural map $\mathscr{F} \to \mathscr{C}^0(\mathscr{U}, \mathscr{F})$, which on an open V is given by $s \mapsto (s_{|U_i\cap V})_i$.

Proposition 4.30. *The sequence* $0 \to \mathscr{F} \to \mathscr{C}^0(\mathscr{U}, \mathscr{F}) \to \mathscr{C}^1(\mathscr{U}, \mathscr{F}) \to \cdots$ *is exact.*

Proposition 4.31. *If* $\mathscr{F}$ *is flasque, then all* $\mathscr{C}^p(\mathscr{U}, \mathscr{F})$ *are flasque, and* $\check{H}^p(\mathscr{U}, \mathscr{F}) = 0$ *for all* $p > 0$.

Proposition 4.32. *For every* i*, there is a natural map* $\check{H}^i(\mathscr{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$*. These maps are compatible with refinements, so we obtain a natural map* $\check{H}^i(X, \mathscr{F}) \to H^i(X, \mathscr{F})$*. These maps are functorial in* $\mathscr{F}$.

Proposition 4.33. *For* $i = 0, 1$*, the natural map* $\check{H}^i(X, \mathscr{F}) \to H^i(X, \mathscr{F})$ *is an isomorphism.*

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(4.13) Vanishing of cohomology of quasi-coherent sheaves on affine schemes.

Theorem 4.34. *Let X be an affine scheme, and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then $\check{H}^i(X, \mathscr{F}) = 0$ for all $i > 0$.*

From this theorem, if follows immediately (using the above results) that $H^1(X, \mathscr{F}) = 0$ for X affine and $\mathscr{F}$ quasi-coherent. In particular, the global section functor on X preserves exactness of every short exact sequence where the left hand term is a quasi-coherent $\mathscr{O}_X$-module. But more is true:

Theorem 4.35. *Let X be an affine scheme, and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.*

This follows from the above using Cartan's Theorem (see e.g., [Go] II Thm. 5.9.2, [Stacks] `01EO`):

Theorem 4.36. *Let X be a ringed space, and let $\mathscr{B}$ be a basis of the topology of X which is stable under finite intersections. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Assume that $\check{H}^i(U, \mathscr{F}) = 0$ for all $i > 0$. Then*
(1) we have $H^i(U, \mathscr{F}) = 0$ for all $i > 0$,
(2) The natural homomorphisms $\check{H}^i(\mathscr{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$ are isomorphisms for all $i \geq 0$ and all covers $\mathscr{U}$ of X consisting of elements of $\mathscr{B}$.
(3) The natural homomorphisms $\check{H}^i(X, \mathscr{F}) \to H^i(X, \mathscr{F})$ are isomorphisms for all $i \geq 0$.

For X noetherian, there is another approach which relies on the following result (see [H] III.3):

Proposition 4.37. *Let A be a noetherian ring, $X = \operatorname{Spec} A$, and let I be an injective A-module. Then $\widetilde{I}$ is a flasque $\mathscr{O}_X$-module.*

From either approach, we also obtain the following consequence (of course, the second approach again works only in the noetherian situation):

Theorem 4.38. *Let X be a separated scheme, and let $\mathscr{U}$ be a cover of X by affine open subschemes. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then the natural homomorphisms $\check{H}^i(\mathscr{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$ are isomorphisms for all $i \geq 0$.*

Corollary 4.39. *Let X be a separated scheme which can be covered by $n+1$ affine open subschemes. Then $H^i(X, \mathscr{F}) = 0$ for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every $i > n$.*

July 10, 2019

(4.14) The cohomology of line bundles on projective space.

References: [H] III.5, [Stacks] `01XS`.

Using Čech cohomology, we can compute the cohomology of line bundles on projective space. It is best to aggregate the results for all $\mathscr{O}(d)$, as we have already seen for their global sections, a result which we repeat as the first statement below.

Theorem 4.40. *Let A be a noetherian ring, $n \geq 1$, $S = [T_0, \dots, T_n]$, $X = \operatorname{Proj}(S) = \mathbb{P}^n_A$. Then*
(1) *There is a natural isomorphism $S \cong \bigoplus_{d\in\mathbb{Z}} H^0(X, \mathscr{O}(d))$.*
(2) *For $i \neq 0, n$ and all $d \in \mathbb{Z}$ we have $H^i(X, \mathscr{O}(d)) = 0$.*
(3) *There is a natural isomorphism $H^n(X, \mathscr{O}(-n-1)) \cong A$.*
(4) *For every r, there is a perfect pairing*

$$H^0(X, \mathscr{O}(r)) \times H^n(X, \mathscr{O}(-r-n-1)) \to H^n(X, \mathscr{O}(-n-1)) \cong A,$$

i.e., a bilinear map which induces isomorphisms

$$H^0(X, \mathscr{O}(r)) \cong H^n(X, \mathscr{O}(-r-n-1))^\vee$$

and

$$H^0(X, \mathscr{O}(r))^\vee \cong H^n(X, \mathscr{O}(-r-n-1))$$

(where $-^vee = \operatorname{Hom}_A(-, A)$ denotes the A-module dual).

(4.15) Finiteness of cohomology of coherent $\mathscr{O}_X$-modules on projective schemes.

Definition 4.41. *Let X be a noetherian scheme. An $\mathscr{O}_X$-module $\mathscr{F}$ is called* coherent*, if it is quasi-coherent and of finite type.*

Let A be a noetherian ring.

Lemma 4.42. *Let $X = \mathbb{P}^n_A$, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then there exist integers $d_1, \dots, d_s$ and a surjective $\mathscr{O}_X$-module homomorphism*

$$\bigoplus_{i=1}^{n} \mathscr{O}(d_i) \twoheadrightarrow \mathscr{F}.$$

Theorem 4.43. *Let X be a projective A-scheme, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then for all $i \geq 0$, the A-module $H^i(X, \mathscr{F})$ is finitely generated.*

At this point it is not hard to prove that higher derived images $R^i f_* \mathscr{F}$ of a coherent $\mathscr{O}_X$-module under a projective morphism $f\colon X \to Y$ are coherent (see [H] III.8).

(4.16) The Theorem of Riemann–Roch revisited.

Reference: [H] III.7, IV.1.

Recall the Theorem of Riemann–Roch that we stated above (Thm. 2.9). In this section, we prove a preliminary version, which also gives a more conceptual view on the "error term" $\dim \Gamma(X, \mathscr{O}(K-D))$ (with notation as above).

Let k be an algebraically closed field.

Definition 4.44. *Let X be a projective k-scheme, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. We call*

$$\chi(\mathscr{F}) = \sum_{i\geq 0} (-1)^i \dim_k H^i(X, \mathscr{F})$$

the Euler characteristic *of $\mathscr{F}$.*

Note that the sum is finite (by the Grothendieck vanishing theorem, Theorem 4.26) and that each term is finite by the results of the previous section.

Now let X/k be a smooth, projective, connected curve. Then $\chi(\mathscr{F}) = \dim_k H^0(X, \mathscr{F}) - \dim_k H^1(X, \mathscr{F})$.

The following theorem is the preliminary version of the Theorem of Riemann–Roch mentioned above.

Theorem 4.45. *Let $\mathscr{L}$ be a line bundle on X. Then*

$$\chi(\mathscr{L}) = \deg(\mathscr{L}) + \chi(\mathscr{O}_X).$$

Now we can define the genus of X as $g := 1 - \chi(\mathscr{O}_X) = \dim_k H^1(X, \mathscr{O}_X)$, and choose for K a divisor with $\mathscr{O}(K) \cong \Omega^1_{X/k}$.

From the above, we immediately get

Corollary 4.46. (Theorem of Riemann) *Let $\mathscr{L}$ be a line bundle on X. Then*

$$\dim_k H^0(X, \mathscr{L}) \geq \deg(\mathscr{L}) + 1 - g.$$

Furthermore, the Theorem of Riemann–Roch will follow from the Serre duality theorem (which will be discussed in the sequel to this course, Algebraic Geometry 3).

Theorem 4.47. (Serre duality) *Let X be a smooth projective curve as above. For every line bundle $\mathscr{L}$ on X, there is a natural isomorphism*

$$H^1(X, \mathscr{L}) \cong H^0(X, \mathscr{L}^{-1} \otimes \Omega^1_{X/k})^\vee$$

of k-vector spaces (where $-^\vee$ denotes the dual k-vector space).

A similar statement holds for every locally free sheaf $\mathscr{L}$ (and the theorem can be vastly further generalized).

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